Example 39. Solve $(x-y)\frac{dy}{dx}=x+y$.

Solution. Divide the DE by x to get $\left(1-\frac{y}{x}\right)\frac{dy}{dx}=1+\frac{y}{x}$. This is a DE of the form $y'=F\left(\frac{y}{x}\right)$.

We therefore substitute $u=\frac{y}{x}$. Then $y=ux$ and $\frac{dy}{dx}=x\frac{du}{dx}+u$.

The resulting DE is $(x-ux)\left(x\frac{du}{dx}+u\right)=x+ux$, which simplifies to $x(1-u)\frac{du}{dx}=1+u^2$.

This DE is separable: $\frac{1-u}{1+u^2}du=\frac{1}{x}dx$

Integrating both sides, we find $\arctan(u)-\frac{1}{2}\ln(1+u^2)=\ln|x|+C$.

Setting $u=y/x$, we get the (general) implicit solution $\arctan(y/x)-\frac{1}{2}\ln(1+(y/x)^2)=\ln|x|+C$.

Comment. We used $\int\frac{1}{1+u^2}du=\arctan(u)+C$ and $\int\frac{x}{1+x^2}dx=\frac{1}{2}\ln(1+x^2)+C$ when integrating.

See Example 35 where we reviewed these integrals.

Example 40. Solve the IVP $\frac{dy}{dx}=2y-3xy^5$, $y(0)=1$.

Solution. This is an example of a Bernoulli equation (with $n=5$). We therefore substitute $u=y^{1-n}=y^{-4}$.

Accordingly, $y=u^{-1/4}$ and, thus, $\frac{dy}{dx}=-\frac{1}{4}u^{-5/4}\frac{du}{dx}$.

The new DE is $-\frac{1}{4}u^{-5/4}\frac{du}{dx}=2u^{-1/4}-3xu^{-5/4}$, which simplifies to $\frac{du}{dx}=-8u+12x$.

This is a linear first-order DE, which we solve according to our recipe:

(a) Rewrite the DE as $\frac{du}{dx}+P(x)u=Q(x)$ with $P(x)=8$ and $Q(x)=12x$.

(b) The integrating factor is $f(x)=\exp\left(\int P(x)dx\right)=e^{8x}$.

(c) Multiply the (rewritten) DE by $f(x)=e^{8x}$ to get

$$\underbrace{e^{8x}\frac{du}{dx}+8e^{8x}u}_{=\frac{d}{dx}[e^{8x}u]}=12xe^{8x}.$$

(d) Integrate both sides to get:

$$e^{8x}u=12\int xe^{8x}dx=12\left(\frac{1}{8}xe^{8x}-\frac{1}{8^2}e^{8x}\right)+C=\frac{3}{2}xe^{8x}-\frac{3}{16}e^{8x}+C$$

Here we used that $\int xe^{ax}dx=\frac{1}{a}xe^{ax}-\frac{1}{a^2}e^{ax}$. (Integration by parts!)

The general solution of the DE for u therefore is $u=\frac{3}{2}x-\frac{3}{16}+Ce^{-8x}$.

Correspondingly, the general solution of the initial DE is $y=u^{-1/4}=1/\sqrt[4]{\frac{3}{2}x-\frac{3}{16}+Ce^{-8x}}$.

Using $y(0)=1$, we find $1=1/\sqrt[4]{C-\frac{3}{16}}$ from which we obtain $C=1+\frac{3}{16}=\frac{19}{16}$.

The unique solution to the IVP therefore is $y=1/\sqrt[4]{\frac{3}{2}x-\frac{3}{16}+\frac{19}{16}e^{-8x}}$.

Solving simple 2nd order DEs

We have the following two useful substitutions for certain simple DEs of order 2:

- $F(y'', y', x) = 0$ (2nd order with "y missing")

  Set $u = y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. Then $y'' = \frac{\mathrm{d}u}{\mathrm{d}x}$. We get the first-order DE $F\left(\frac{\mathrm{d}u}{\mathrm{d}x}, u, x\right) = 0$.

- $F(y'', y', y) = 0$ (2nd order with "x missing")

  Set $u = y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. Then $y'' = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot u$. We get the first-order DE $F\left(u\frac{\mathrm{d}u}{\mathrm{d}y}, u, y\right) = 0$.

Example 41. Solve $y'' = x - y'$.

Solution. We substitute $u = y'$, which results in the first-order DE $u' = x - u$.

This DE is linear and, using our recipe (see below for the details), we can solve it to find $u = x - 1 + Ce^{-x}$.

Since $y' = u$, we conclude that the general solution is $y = \int (x - 1 + Ce^{-x})\mathrm{d}x = \frac{1}{2}x^2 - x - Ce^{-x} + D$.

Important comment. This is a DE of order 2. Hence, as expected, the general solution has two free parameter.

Solving the linear DE. To solve $u' = x - u$ (also see Example 31, where we had solved this DE before), we

(a) rewrite the DE as $\frac{\mathrm{d}u}{\mathrm{d}x} + P(x)u = Q(x)$ with $P(x) = 1$ and $Q(x) = x$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^x$.

(c) Multiply the (rewritten) DE by $f(x) = e^x$ to get $\underbrace{e^x\frac{\mathrm{d}u}{\mathrm{d}x} + e^x u}_{= \frac{\mathrm{d}}{\mathrm{d}x}[e^x u]} = x e^x$.

(d) Integrate both sides to get (using integration by parts): $e^x u = \int x e^x \mathrm{d}x = x e^x - e^x + C$

Hence, the general solution of the DE for u is $u = x - 1 + Ce^{-x}$, which is what we used above.

Example 42. (homework) Solve the IVP $y'' = x - y'$, $y(0) = 1$, $y'(0) = 2$.

Solution. As in the previous example, we find that the general solution to the DE is $y(x) = \frac{1}{2}x^2 - x - Ce^{-x} + D$.

Using $y'(x) = x - 1 + Ce^{-x}$ and $y'(0) = 2$, we find that $2 = -1 + C$. Hence, $C = 3$.

Then, using $y(x) = \frac{1}{2}x^2 - x - 3e^{-x} + D$ and $y(0) = 1$, we find $1 = -3 + D$. Hence, $D = 4$.

In conclusion, the unique solution to the IVP is $y(x) = \frac{1}{2}x^2 - x - 3e^{-x} + 4$.

Example 43. (extra) Find the general solution to $y'' = 2yy'$.

Solution. We substitute $u = y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. Then $y'' = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot u$.

Therefore, our DE turns into $u\frac{\mathrm{d}u}{\mathrm{d}y} = 2yu$.

Dividing by u, we get $\frac{\mathrm{d}u}{\mathrm{d}y} = 2y$. [Note that we lose the solution $u = 0$, which gives the singular solution $y = C$.]

Hence, $u = y^2 + C$. It remains to solve $y' = y^2 + C$. This is a separable DE.

$\frac{1}{C + y^2}\mathrm{d}y = \mathrm{d}x$. Let us restrict to $C = D^2 \geqslant 0$ here. (This means we will only find "half" of the solutions.)

$\int \frac{1}{D^2 + y^2}\mathrm{d}y = \frac{1}{D^2}\int \frac{1}{1 + (y/D)^2}\mathrm{d}y = \frac{1}{D}\arctan(y/D) = x + A$.

Solving for y, we find $y = D\tan(Dx + AD) = D\tan(Dx + B)$. $[B = AD]$

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