Review. We can solve linear first-order DEs using **integrating factors**.

First, put the DE in standard form $y' + P(x)y = Q(x)$. Then $f(x) = \exp\left(\int P(x)\mathrm{d}x\right)$ is the integrating factor.

The key is that we get on the left-hand side $f(x)y' + f(x)P(x)y = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)\,y]$. We can therefore integrate both sides with respect to x (the right-hand side is $f(x)Q(x)$ which is just a function depending on x—not y!).

Example 32. Solve $x^2\,y' = 1 - xy + 2x$, $y(1) = 3$.

Solution. This is a linear first-order DE. We can therefore solve it according to the recipe above.

(a) Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ (standard form) with $P(x) = \frac{1}{x}$ and $Q(x) = \frac{1}{x^2} + \frac{2}{x}$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^{\ln x} = x$.

Here, we could write $\ln x$ instead of $\ln|x|$ because the initial condition tells us that $x > 0$, at least locally.
Comment. We can also choose a different constant of integration but that would only complicate things.

(c) Multiply the DE (in standard form) by $f(x) = x$ to get

$$\underbrace{x\frac{\mathrm{d}y}{\mathrm{d}x} + y}_{=\frac{\mathrm{d}}{\mathrm{d}x}[xy]} = \frac{1}{x} + 2.$$

(d) Integrate both sides to get (again, we use that $x > 0$ to avoid having to use $|x|$)

$$xy = \int\left(\frac{1}{x} + 2\right)\mathrm{d}x = \ln x + 2x + C.$$

Using $y(1) = 3$ to find C, we get $1 \cdot 3 = \ln(1) + 2 \cdot 1 + C$ which results in $C = 3 - 2 = 1$.
Hence, the (unique) solution to the IVP is $y = \frac{\ln(x) + 2x + 1}{x}$.

Example 33. Solve $xy' = 2y + 1$, $y(-2) = 0$.

Solution. This is a linear first-order DE.

(a) Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ (standard form) with $P(x) = -\frac{2}{x}$ and $Q(x) = \frac{1}{x}$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^{-2\ln|x|} = e^{-2\ln(-x)} = (-x)^{-2} = \frac{1}{x^2}$.

Here, we used that, at least locally, $x < 0$ (because the initial condition is $x = -2 < 0$) so that $|x| = -x$.

(c) Multiply the DE (in standard form) by $f(x) = \frac{1}{x^2}$ to get

$$\underbrace{\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2}{x^3}y}_{=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x^2}y\right]} = \frac{1}{x^3}.$$

(d) Integrate both sides to get

$$\frac{1}{x^2}y = \int\frac{1}{x^3}\mathrm{d}x = -\frac{1}{2x^2} + C.$$

Hence, the general solution is $y(x) = -\frac{1}{2} + Cx^2$.
Solving $y(-2) = -\frac{1}{2} + 4C = 0$ for C yields $C = \frac{1}{8}$. Thus, the (unique) solution to the IVP is $y(x) = \frac{1}{8}x^2 - \frac{1}{2}$.

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Example 34. (extra) Solve $y' = 2y + 3x - 1$, $y(0) = 2$.

Solution. This is a linear first-order DE.

(a) Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ (standard form) with $P(x) = -2$ and $Q(x) = 3x - 1$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^{-2x}$.

(c) Multiply the DE (in standard form) by $f(x) = e^{-2x}$ to get

$$\underbrace{e^{-2x}\frac{\mathrm{d}y}{\mathrm{d}x} - 2e^{-2x}y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[e^{-2x}y]} = (3x-1)e^{-2x}.$$

(d) Integrate both sides to get

$$\begin{aligned} e^{-2x}y &= \int (3x-1)e^{-2x}\mathrm{d}x \\ &= 3\int x e^{-2x}\mathrm{d}x - \int e^{-2x}\mathrm{d}x \\ &= 3\left(-\frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x}\right) - \left(-\frac{1}{2}e^{-2x}\right) + C \\ &= -\frac{3}{2}x e^{-2x} - \frac{1}{4}e^{-2x} + C. \end{aligned}$$

Here, we used that $\int x e^{-2x}\mathrm{d}x = -\frac{1}{2}x e^{-2x} + \frac{1}{2}\int e^{-2x}\mathrm{d}x = -\frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x}$ (for instance, via integration by parts with $f(x) = x$ and $g'(x) = e^{-2x}$).

Hence, the general solution is $y(x) = -\frac{3}{2}x - \frac{1}{4} + Ce^{2x}$.

Solving $y(0) = -\frac{1}{4} + C = 2$ for C yields $C = \frac{9}{4}$.

In conclusion, the (unique) solution to the IVP is $y(x) = -\frac{3}{2}x - \frac{1}{4} + \frac{9}{4}e^{2x}$.