Review. Existence and uniqueness theorem (Theorem 23) for an IVP $y' = f(x, y)$, $y(a) = b$:
If $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous around (a, b) then, locally, the IVP has a unique solution.

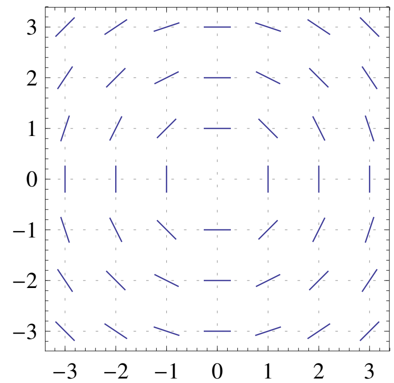
Example 29. Consider, again, the IVP $y' = -x / y$, $y(a) = b$. Discuss existence and uniqueness of solutions (without solving).

Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = -x / y$.

We compute that $\frac{\partial}{\partial y} f(x, y) = x / y^2$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $y \neq 0$.

Hence, if $b \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.



Comment. In Example 14, we found that the DE $y' = -x / y$ is solved by $y(x) = \pm\sqrt{D - x^2}$.

Assume $b > 0$ (things work similarly for $b < 0$). Then $y(x) = \sqrt{D - x^2}$ solves the IVP (we need to choose D so that $y(a) = b$) if we choose $D = a^2 + b^2$. This confirms that there exists a solution. On the other hand, uniqueness means that there can be no other solution to the IVP than this one.

What happens in the case $b = 0$?

Solution. In this case, the existence and uniqueness theorem does not guarantee anything. If $a \neq 0$, then $y(x) = \sqrt{a^2 - x^2}$ and $y(x) = -\sqrt{a^2 - x^2}$ both solve the IVP (so we certainly don't have uniqueness), however only in a weak sense: namely, both of these solutions are not valid locally around $x = a$ but only in an interval of which a is an endpoint (for instance, the IVP $y' = -x / y$, $y(2) = 0$ is solved by $y(x) = \pm\sqrt{4 - x^2}$ but both of these solutions are only valid on the interval $[-2, 2]$ which ends at 2, and neither of these solutions can be extended past 2).

Linear first-order DEs

A **linear differential equation** is one where the function y and its derivatives only show up linearly (i.e. there are no terms such as y^2, $1 / y$, $\sin(y)$ or $y \cdot y'$).

As such, the most general linear first-order DE is of the form

$$A(x)y' + B(x)y + C(x) = 0.$$

Such a DE can be rewritten in the following "**standard form**" by dividing by $A(x)$ and rearranging:

(linear first-order DE in standard form)

$$y' + P(x)y = Q(x)$$

We will use this standard form when solving linear first-order DEs.

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Example 30. (extra "warmup") Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2$.

Solution. (separation of variables) $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, $-\frac{1}{y} = x^2 + C$.

Hence the general solution is $y = \frac{1}{D - x^2}$. [There also is the singular solution $y = 0$.]

Solution. (in other words) Note that $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ can be written as $\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{1}{y}\right] = \frac{\mathrm{d}}{\mathrm{d}x}[x^2]$.
From there it follows that $-\frac{1}{y} = x^2 + C$, as above.

We now use the idea of writing both sides as a derivative (which we then integrate!) to also solve DEs that are not separable. We will be able to handle all first-order linear DEs this way.

The multiplication by $\frac{1}{y^2}$ will be replaced by multiplication with a so-called **integrating factor**.

Example 31. Solve $y' = x - y$.

Comment. Note that we cannot use separation of variables this time.

Solution. Rewrite the DE as $y' + y = x$.
Next, multiply both sides with e^x (we will see in a little bit how to find this "integrating factor") to get

$$\underbrace{e^x y' + e^x y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[e^x y]} = xe^x.$$

The "magic" part is that we are able to realize the new left-hand side as a derivative!
We can then integrate both sides to get

$$e^x y = \int xe^x \mathrm{d}x = xe^x - e^x + C.$$

From here it follows that $y = x - 1 + Ce^{-x}$.
Comment. For the final integral, we used that $\int xe^x \mathrm{d}x = xe^x - \int e^x \mathrm{d}x = xe^x - e^x + C$ which follows, for instance, via integration by parts (with $f(x) = x$ and $g'(x) = e^x$ in the formula reviewed below).

Review. The multiplication rule $(fg)' = f'g + fg'$ implies $fg = \int f'g + \int fg'$.

The latter is equivalent to **integration by parts**:

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x$$

Comment. Sometimes, one writes $g'(x)\mathrm{d}x = \mathrm{d}g(x)$.

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In general, we can solve any **linear first-order DE** $y' + P(x)y = Q(x)$ in this way.

- We want to multiply with an **integrating factor** $f(x)$ such that the left-hand side of the DE becomes

$$f(x)y' + f(x)P(x)y = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)y].$$

  Since $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)y] = f(x)y' + f'(x)y$, we need $f'(x) = f(x)P(x)$ for that.

- Check that $f(x) = \exp\left(\int P(x)\mathrm{d}x\right)$ has this property.

  Comment. This follows directly from computing the derivative of this $f(x)$ via the chain rule.

  Homework. On the other hand, note that finding f meant solving the DE $f' = P(x)\,f$. This is a separable DE. Solve it by separation of variables to arrive at the above formula for $f(x)$ yourself.

  Just to make sure. There is no difference between $\exp(x)$ and e^x. Here, we prefer the former notation for typographical reasons.

With that integrating factor, we have the following recipe for solving any linear first-order equation:

(solving linear first-order DEs)

(a) Write the DE in the **standard form** $y' + P(x)y = Q(x)$.

(b) Compute the **integrating factor** as $f(x) = \exp\left(\int P(x)\mathrm{d}x\right)$.

[We can choose any constant of integration.]

(c) Multiply the DE from part (a) by $f(x)$ to get

$$\underbrace{f(x)y' + f(x)P(x)y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[f(x)y]} = f(x)Q(x).$$

(d) Integrate both sides to get

$$f(x)y = \int f(x)Q(x)\mathrm{d}x + C.$$

Then solve for y by dividing by $f(x)$.

Comment. For better understanding, we prefer to go through the above steps. On the other hand, we can combine these steps into the following formula for the general solution of $y' + P(x)y = Q(x)$:

$$y = \frac{1}{f(x)}\left(\int f(x)Q(x)\mathrm{d}x + C\right) \quad \text{where } f(x) = e^{\int P(x)\mathrm{d}x}$$

Existence and uniqueness. Note that the solution we construct exists on any interval on which P and Q are continuous (not just on some possibly very small interval). This is better than what the existence and uniqueness theorem (Theorem 23) can guarantee. This is one of the many ways in which linear DEs have particularly nice properties compared to DEs in general.