In the following example, we first proceed like we did when producing a slope field to compute slopes (and, therefore, tangent lines) of solutions. Indeed, besides the slope y', we can further compute further derivatives like y'' or y''' by differentiating the DE.

Do you recall how y'' tells us about the curvature of a function $y(x)$?

Example 16. Consider the DE $x^2 y' = 1 + x y^3$. Suppose that $y(x)$ is a solution passing through the point $(2, 1)$.

Important. This is the same as saying that $y(x)$ solves the IVP $x^2 y' = 1 + x y^3$ with $y(2) = 1$.

(a) Determine $y'(2)$.

(b) Determine the tangent line of $y(x)$ at $(2, 1)$.

(c) Determine $y''(2)$.

Comment. Note that this DE is not separable.

Solution.

(a) At the point $(2, 1)$ we have $x = 2$ and $y = 1$. Plugging these values into the differential equation, we get $4y' = 1 + 2 \cdot 1^3 = 3$ which we can solve for y' to find $y' = \frac{3}{4}$.

Since y' is short for $y'(x) = y'(2)$, we have found $y'(2) = \frac{3}{4}$.

(b) The tangent line is the line through $(2, 1)$ with slope $\frac{3}{4}$ (computed in the previous part).

From this information, we can immediately write down its equation in the form $y = \frac{3}{4}(x - 2) + 1$.

(c) To get our hands on $y''(2)$, we can differentiate (with respect to x) both sides of $x^2 y' = 1 + x y^3$. Applying the product rule, we have $\frac{\mathrm{d}}{\mathrm{d}x} x^2 y'(x) = 2x y'(x) + x^2 y''(x) = 2x y' + x^2 y''$ as well as $\frac{\mathrm{d}}{\mathrm{d}x}(1 + x y(x)^3) = y(x)^3 + x \cdot 3y(x)^2 \cdot y'(x) = y^3 + 3x y^2 y'$.
Thus $2x y' + x^2 y'' = y^3 + 3x y^2 y'$. To find $y''(2)$, we plug in $x = 2$, $y = 1$, $y' = \frac{3}{4}$.

This results in $2 \cdot 2 \cdot \frac{3}{4} + 4y'' = 1 + 3 \cdot 2 \cdot 1 \cdot \frac{3}{4}$ or $3 + 4y'' = \frac{11}{2}$. It follows that $y'' = \frac{1}{4} \cdot \frac{5}{2} = \frac{5}{8}$.

Comment. Alternatively, we can rewrite the DE as $y' = \frac{1}{x^2} + \frac{1}{x} y^3$ and then differentiate. Do it!

Comment. Do you recall from Calculus what it means visually to have $y'' = \frac{5}{8}$?

[Since $y'' > 0$ it means that our function is concave up at $(2, 1)$. As such, its graph will lie above the tangent line.]

Comment. Note that we could continue and likewise find $y'''(2)$ or higher derivatives at $x = 2$. This is the starting point for the power series method typically discussed in Differential Equations II.

Solving DEs: Separation of variables, cont'd

In general, **separation of variables** solves $y' = g(x) h(y)$ by writing the DE as $\frac{1}{h(y)} \mathrm{d}y = g(x) \mathrm{d}x$.

Note that $\frac{1}{h(y)} \frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$ is indeed equivalent to $\int \frac{1}{h(y)} \mathrm{d}y = \int g(x) \mathrm{d}x + C$. Why?! (Apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to the integrals...)

Example 17. Solve the IVP $y' = -\frac{x}{y}$, $y(0) = -3$.

Comment. Instead of using what we found earlier in Example 14, we start from scratch to better illustrate the solution process (and how we can use the initial condition right away to determine the value of the constant of integration).

Solution. We separate variables to get $y\,\mathrm{d}y = -x\,\mathrm{d}x$.

Integrating gives $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$, and we use $y(0) = -3$ to find $\frac{1}{2}(-3)^2 = 0 + C$ so that $C = \frac{9}{2}$.

Hence, $x^2 + y^2 = 9$ is an **implicit** form of the solution.

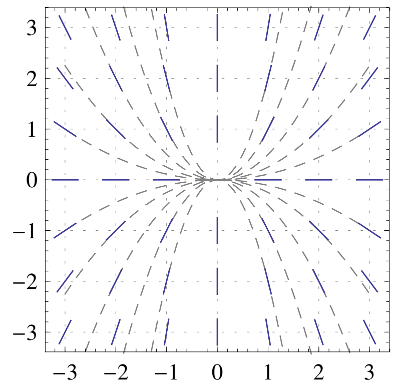
Solving for y, we get $y = -\sqrt{9 - x^2}$ (note that we have to choose the negative sign so that $y(0) = -3$).

Comment. Note that our solution is a **local solution**, meaning that it is valid (and solves the DE) locally around $x = 0$ (from the initial condition). However, it is not a **global solution** because it doesn't make sense outside of x in the interval $[-3, 3]$.

Example 18. Consider the DE $xy' = 2y$.

Sketch its slope field.

Challenge. Try to guess solutions $y(x)$ from the slope field.



Solution. For instance, to find the slope at the point $(3, 1)$, we plug $x = 3$, $y = 1$ into the DE to get $3y' = 2$. Hence, the slope is $y' = 2/3$.

The resulting slope field is sketched on the right.

Solution of the challenge. Trace out the solution through $(1, 1)$ (and then some other points). Their shape looks like a parabola, so that we might guess that $y(x) = Cx^2$ solves the DE. Check that this is indeed the case by plugging into the DE!

Example 19. Solve the IVP $xy' = 2y$, $y(1) = 2$.

Solution. Rewrite the DE as $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x}$.

Then multiply both sides with $\mathrm{d}x$ and integrate both of them to get $\int \frac{1}{y}\mathrm{d}y = \int \frac{2}{x}\,\mathrm{d}x$.

Hence, $\ln|y| = 2\ln|x| + C$.

The initial condition $y(1) = 2$ tells us that, at least locally, $x > 0$ and $y > 0$. Thus $\ln(y) = 2\ln(x) + C$.

Moreover, plugging in $x = 1$ and $y = 2$, we find $C = \ln(2)$.

Solving $\ln(y) = 2\ln(x) + \ln(2)$ for y, we find $y = e^{2\ln(x) + \ln(2)} = 2x^2$.

Comment. When solving a DE or IVP, we can generally only expect to find a **local solution**, meaning that our solution might only be valid in a small interval around the initial condition (here, we can only expect $y(x)$ to be a solution for all x in an interval around 1; especially since we assumed $x > 0$ in our solution). However, we can check (do it!) that the solution $y = 2x^2$ is actually a **global solution** (meaning that it is a solution for all x, not just locally around 1).

Let's solve the same differential equation with a different choice of initial condition:

Example 20. Solve the IVP $xy' = 2y$, $y(1) = -1$.

Solution. Again, we rewrite the DE as $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x}$, multiply both sides with $\mathrm{d}x$, and integrate to get $\int \frac{1}{y}\mathrm{d}y = \int \frac{2}{x}\,\mathrm{d}x$.

Hence, $\ln|y| = 2\ln|x| + C$. The initial condition $y(1) = -1$ tells us that, at least locally, $x > 0$ and $y < 0$ (note that this means $|y| = -y$). Thus $\ln(-y) = 2\ln(x) + C$.

Moreover, plugging in $x = 1$ and $y = -1$, we find $C = 0$.

Solving $\ln(-y) = 2\ln(x)$ for y, we find $y = -e^{2\ln(x)} = -x^2$. We easily verify that this is indeed a global solution.

Example 21. $y' = x + y$ is a DE for which the variables cannot be separated.

No worries, very soon we will have several tools to solve this DE as well.

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