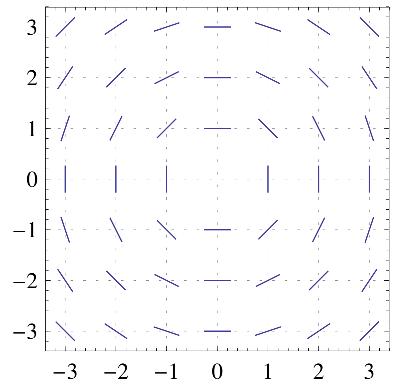
Slope fields, or sketching solutions to DEs

Example 13. Consider the DE $y' = -x / y$.

Let's pick a point, say, $(1, 2)$. If a solution $y(x)$ is passing through that point, then its slope has to be $y' = -1 / 2$. We therefore draw a small line through the point $(1, 2)$ with slope $-1 / 2$. Continuing in this fashion for several other points, we obtain the **slope field** on the right.



With just a little bit of imagination, we can now anticipate the solutions to look like (half)circles around the origin. Let us check whether $y(x) = \sqrt{r^2 - x^2}$ might indeed be a solution!

$y'(x) = \frac{1}{2} \frac{-2x}{\sqrt{r^2 - x^2}} = -x / y(x)$. So, yes, we actually found solutions!

Solving DEs: Separation of variables

Example 14. Solve the DE $y' = -\frac{x}{y}$.

Solution. Rewrite the DE as $\frac{dy}{dx} = -\frac{x}{y}$.

Separate the variables to get $y\,dy = -x\,dx$ (in particular, we are multiplying both sides by dx).

Integrating both sides, we get $\int y\,dy = \int -x\,dx$.

Computing both integrals results in $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$ (we combine the two constants of integration into one).

Hence $x^2 + y^2 = D$ (with $D = 2C$).

This is an **implicit form** of the solutions to the DE. We can make it explicit by solving for y. Doing so, we find $y(x) = \pm\sqrt{D - x^2}$ (choosing $+$ gives us the upper half of a circle, while the negative sign gives us the lower half).

Comment. The step above where we break $\frac{dy}{dx}$ apart and then integrate may sound sketchy!

However, keep in mind that, after we find a solution $y(x)$, even if by sketchy means, we can (and should!) verify that $y(x)$ is indeed a solution by plugging into the DE. We actually already did that in the previous example!

Example 15. (extra)

Comment. In this example, we use $x(t)$ instead of $y(x)$ for the function described by the differential equation. In general, of course, any choice of variable names is possible. If we write something like x' or y' it needs to be clear from the context with respect to which variable that derivative is meant (such as $x' = \frac{d}{dt}x(t)$).

(a) Solve the DE $\frac{dx}{dt} = kx^2$.

(b) Verify your answer from the first part.

(c) Solve the IVP $\frac{dx}{dt} = kx^2$, $x(0) = 2$.

(d) Solve the IVP $\frac{dx}{dt} = kx^2$, $x(0) = 0$.

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Solution.

(a) This DE is separable: $\frac{1}{x^2}\mathrm{d}x = k\,\mathrm{d}t$. Integrating, we find $-\frac{1}{x} = kt + B$. (We plan to replace B by a new constant C in a moment.) Hence, $x = -\frac{1}{kt+B} = \frac{1}{C-kt}$.

[Here, $C = -B$ but that relationship doesn't matter; it only matters that the solution has a free parameter.]

Comment. Note that we did not find the solution $x = 0$ (lost when dividing by x^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). [Although, we can obtain it from the general solution by letting $C \to \infty$.]

See the last part for a case when this "missing" solution is needed.

(b) Starting with $x(t) = \frac{1}{C-kt}$, we compute that $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{(C-kt)^2}\cdot(-k) = \frac{k}{(C-kt)^2}$.

On the other hand, $kx^2 = k\left(\frac{1}{C-kt}\right)^2 = \frac{k}{(C-kt)^2}$. Since this matches what we got for $\frac{\mathrm{d}x}{\mathrm{d}t}$, it is indeed true that $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$.

(c) We start with $x(t) = \frac{1}{C-kt}$ (which we know solves the DE for any value of C) and seek to choose C so that $x(0) = 2$.

Since $x(0) = \left[\frac{1}{C-kt}\right]_{t=0} = \frac{1}{C} \overset{!}{=} 2$, we find $C = \frac{1}{2}$.

Hence, the IVP has the (unique) solution $x(t) = \frac{1}{1/2 - kt}$.

(d) Proceeding as in the previous part, we now arrive at the impossible equation $\frac{1}{C} \overset{!}{=} 0$.

However, this suggests that we should consider taking $C \to \infty$ in $x(t) = \frac{1}{C-kt}$, which results in $x(t) = 0$.

Indeed, it is easy to verify (make sure you know what this entails!) that $x(t) = 0$ solves the IVP.