Example 6. (review) Solve the DE $y' = x^2 + x$.

Solution. Note that the DE simply asks for a function $y(x)$ with a specific derivative (in particular, the right-hand side does not involve $y(x)$). In other words, the desired $y(x)$ is an **antiderivative** of $x^2 + x$. We know from Calculus II that we can find antiderivatives by integrating:

$$y(x) = \int (x^2 + x)\mathrm{d}x = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

Moreover, we know from Calculus II that there are no other solutions. In other words, we found the **general solution** to the DE.

If the highest derivative appearing in a DE is an rth derivative, we say that the DE has **order** r.

For instance. The DE $y' = 3\sqrt{1 - y^2}$ has order 1 (such DEs are also called first order DEs).
On the other hand, the DE $y'' = y' + 6y$ has order 2 (such DEs are also called second order DEs).

As we will observe in the next few examples, we typically expect that the general solution of a DE of **order** r has r **parameters** (or degrees of freedom).

A first initial value problem

To single out a **particular solution**, we need to specify additional conditions (typically one condition per parameter in the general solution). For instance, it is common to impose **initial conditions** such as $y(1) = 2$. A DE together with an initial condition is called an **initial value problem** (IVP).

Example 7. Solve the IVP $y' = x^2 + x$ with $y(1) = 2$.

Solution. From the previous example, we know that $y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$.
Since $y(1) = \frac{1}{3} + \frac{1}{2} + C = \frac{5}{6} + C \overset{!}{=} 2$, we find $C = 2 - \frac{5}{6} = \frac{7}{6}$.
Hence, $y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{7}{6}$ is the (unique) solution of the IVP.

Example 8. (homework) Solve the DE $y'' = x^2 + x$.

Solution. We now take two antiderivatives of $x^2 + x$ to get

$$y(x) = \int\int (x^2 + x)\mathrm{d}x\mathrm{d}x = \int \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + C\right)\mathrm{d}x = \frac{1}{12}x^4 + \frac{1}{6}x^3 + Cx + D,$$

where it is important that we give the second constant of integration a name different from the first.

Important comment. This is the general solution to the DE. The DE is of order 2 and, as expected, the general solution has 2 parameters.

Verifying if a function solves a DE

Given a function, we can always check whether it solves a DE!

We can just plug it into the DE and see if left and right side agree. This means that we can always check our work as well as that we can verify solutions generated by someone else (or a computer algebra system) even if we don't know the techniques for solving the DE.

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Example 9. (warmup) Consider the DE $y'' = y' + 6y$.

(a) Is $y(x) = e^{2x}$ a solution?

(b) Is $y(x) = e^{3x}$ a solution?

Solution.

(a) We compute $y' = 2e^{2x}$ and $y'' = 4e^{2x}$.
Since $y' + 6y = 8e^{2x}$ is different from $y'' = 4e^{2x}$, we conclude that $y(x) = e^{2x}$ is not a solution.

(b) We compute $y' = 3e^{3x}$ and $y'' = 9e^{3x}$.
Since $y' + 6y = 9e^{3x}$ is equal to $y'' = 9e^{3x}$, we conclude that $y(x) = e^{3x}$ is a solution of the DE.

We will soon be able to completely solve differential equations such as in the previous example. The following gives a taste of how we will go about it:

Example 10. (cont'd) Consider the DE $y'' = y' + 6y$. For which r is e^{rx} a solution?

Solution. If $y(x) = e^{rx}$, then $y'(x) = r e^{rx}$ and $y''(x) = r^2 e^{rx}$.
Plugging $y(x) = e^{rx}$ into the DE, we get $r^2 e^{rx} = r e^{rx} + 6e^{rx}$ which simplifies to $r^2 = r + 6$.
This has the two solutions $r = -2$, $r = 3$. Hence e^{-2x} and e^{3x} are solutions of the DE.

In fact, we check that $Ae^{-2x} + Be^{3x}$ is a **two-parameter family** of solutions to the DE.

Important comment. It is no coincidence that the order of the DE is 2, whereas the previous example has order 1. In general, we expect a DE of order r to have a solution with r parameters.

Example 11. Consider the DE $e^y y' = 1$.

(a) Is $y(x) = \ln(x)$ a solution to the DE?

(b) Is $y(x) = \ln(x) + C$ a solution to the DE?

(c) Is $y(x) = \ln(x + C)$ a solution to the DE?

Solution.

(a) Since $y'(x) = \frac{1}{x}$ and $e^{y(x)} = e^{\ln(x)} = x$, we have $e^y y' = x \cdot \frac{1}{x} \stackrel{\checkmark}{=} 1$.
Hence, $y(x) = \ln(x)$ is a solution to the given DE.

(b) Since $y'(x) = \frac{1}{x}$ and $e^{y(x)} = e^{\ln(x) + C} = x e^C$, we have $e^y y' = x e^C \cdot \frac{1}{x} = e^C$. Thus the DE is satisfied only if $e^C = 1$ which only happens if $C = 0$ (which is the case in the first part).
Hence, $y(x) = \ln(x) + C$ is not a solution to the given DE except if $C = 0$.

(c) Since $y'(x) = \frac{1}{x + C}$ and $e^{y(x)} = e^{\ln(x + C)} = x + C$, we have $e^y y' = (x + C) \cdot \frac{1}{x + C} \stackrel{\checkmark}{=} 1$.
Hence, $y(x) = \ln(x + C)$ is indeed a one-parameter family of solutions to the given DE.

Any fixed function usually solves many DEs

Usually, we start with a DE (which comes, for instance, from physical laws) and want to solve it. In the next example, we start with a function and determine several DEs that it solves.

Example 12. Determine several (random) DEs that $y(x) = \sin(3x)$ solves.

Solution. Here are some options (but there are many more):

(a) We compute $y'(x) = 3\cos(3x)$. Accordingly, $y(x) = \sin(3x)$ solves the DE $y' = 3\cos(3x)$.

Comment. This, however, is not an "interesting" choice. In particular, this DE could be simply solved by computing an antiderivative (as in the previous examples).

Comment. Note that there are further solutions to this DE: the **general solution** is $\int 3\cos(3x)\mathrm{d}x = \sin(3x) + C$ where C is any constant. We say that $y(x) = \sin(3x) + C$ is a **one-parameter family** of solutions to the DE. C is called a **degree of freedom**.

(b) Note that $y'(x) = 3\cos(3x) = 3\sqrt{1 - (\sin(3x))^2} = 3\sqrt{1 - y(x)^2}$ (for x close to 0).

[Here we used that $\cos(x)^2 + \sin(x)^2 = 1$, which implies that $\cos(x) = \sqrt{1 - \sin(x)^2}$.]

Hence, $y(x) = \sin(3x)$ solves the differential equation $y' = 3\sqrt{1 - y^2}$.

Comment. In the above, we restrict x to $\left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$ so that $\cos(3x) > 0$. Less precisely, we can say that x is close to 0. (It is a common feature of DEs that we work with values of x close to a certain initial value.)

(c) We compute $y''(x) = -9\sin(3x)$. Accordingly, $y(x) = \sin(3x)$ solves the DE $y'' = -9\sin(3x)$.

Comment. Once more this DE is easy (because it only involves y'' but not y or y'). Hence, we can find the general solution by simply taking two antiderivatives:

$$y(x) = \iint -9\sin(3x)\mathrm{d}x\,\mathrm{d}x = \int (3\cos(3x) + C)\mathrm{d}x = \sin(3x) + Cx + D.$$

It is important that we give the second constant of integration a name different from the first. That way, we see that the general solution has 2 degrees of freedom. This matches the fact that the order of the DE is 2.

Important comment. This is no coincidence. In general, we expect a DE of order r to have a general solution with r parameters.

(d) $y(x) = \sin(3x)$ also solves the DE $y'' = -9y$.

Comment. This is again a DE of order 2. Therefore the general solution should have 2 degrees of freedom. Later we will learn to solve such DEs. For now, we can verify that $y(x) = A\sin(3x) + B\cos(3x)$ is a solution for any constants A and B.

Homework. Check that $y(x) = \sin(3x) + C$ does not solve the DE $y'' = -9y$.

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