Midterm #2

MATH 238 — Differential Equations I
Wednesday, Nov 15, 2023

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 33 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (10 points) Determine the general solution of the system $\begin{array}{rcl} y_1' & = & y_1 + y_2 \\ y_2' & = & 3y_1 - y_2 + 6e^x \end{array}$.

Solution. Using $y_2 = y_1' - y_1$ (from the first equation) in the second equation, we get $y_1'' - y_1' = 3y_1 - (y_1' - y_1) + 6e^x$. Simplified, this is $y_1'' - 4y_1 = 6e^x$. This is an inhomogeneous linear DE with constant coefficients. Since the "old" roots are ± 2, while the "new" root is 1, there must a particular solution of the form $y_1 = Ae^x$. For this y_1, $y_1'' - 4y_1 = (1-4)Ae^x = -3Ae^x \overset{!}{=} 6e^x$. Hence, $A = -2$ and the particular solution is $y_1 = -2e^x$. The corresponding general solution is $y_1 = -2e^x + C_1e^{2x} + C_2e^{-2x}$.

Correspondingly, $y_2 = -2e^x + 2C_1e^{2x} - 2C_2e^{-2x} - (-2e^x + C_1e^{2x} + C_2e^{-2x}) = C_1e^{2x} - 3C_2e^{-2x}$.

Problem 2. (5 points) The position $y(t)$ of a certain mass on a spring is described by $2y'' + dy' + 4y = F\sin(\omega t)$.

(a) Assume first that there is no external force, i.e. $F = 0$. For which values of d is the system underdamped?

(b) Now, $F \neq 0$ and the system is undamped, i.e. $d = 0$. For which values of ω, if any, does resonance occur?

Solution.

(a) The discriminant of the characteristic equation is $d^2 - 32$. Hence the system is underdamped if $d^2 - 32 < 0$, that is $d < \sqrt{32} = 4\sqrt{2}$.

(b) The natural frequency is $\sqrt{2}$. Resonance therefore occurs if $\omega = \sqrt{2}$.

Problem 3. (5 points) Let L be a linear differential operator of order 4 with constant real coefficients. Suppose that $1 - 2i$ is a repeated characteristic root of L.

(a) What is the general solution to $Ly = 0$?

(b) Write down the simplest form of a particular solution y_p of the DE $Ly = 2e^x\cos(2x) - 5xe^x$ with undetermined coefficients.

Solution. Since L is real, if $1 - 2i$ is a repeated characteristic root of L, then $1 + 2i$ must be a repeated characteristic root of L as well. Hence, the 4 characteric roots ("old" roots) must be $1 \pm 2i, 1 \pm 2i$.

(a) The general solution is $(C_1 + C_2x)e^x\cos(2x) + (C_3 + C_4x)e^x\sin(2x)$.

(b) The "new" roots are $1 \pm 2i, 1, 1$.

Hence, there must a particular solution of the form $C_1x^2e^x\cos(2x) + C_2x^2e^x\sin(2x) + (C_3 + C_4x)e^x$.

Problem 4. (3 points) Write the (third-order) differential equation $y''' + 7y'' - 5y' - 2y = \cos(2x)$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, the DE translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = 2y_1 + 5y_2 - 7y_3 + \cos(2x) \end{cases}$.

In matrix form, with $\boldsymbol{y} = (y_1, y_2, y_3)$, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 5 & -7 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 0 \\ 0 \\ \cos(2x) \end{bmatrix}$.

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Problem 5. (6 points) The mixtures in two tanks T_1, T_2 are kept uniform by stirring. Brine containing 3 lb of salt per gallon enters T_1 at a rate of 3 gal/min, while brine containing 2 lb of salt per gallon enters T_2 at a rate of 4gal/min. Mixed solution from T_1 is pumped into T_2 at a rate of 1 gal/min, and also from T_2 into T_1 at a rate of 2 gal/min. Each tank initially contains 10 gal of pure water.

Denote by $y_i(t)$ the amount (in pounds) of salt in tank T_i at time t (in minutes). Derive a system of linear differential equations for the y_i, including initial conditions. (Do *not* attempt to solve the system.)

Solution. Note that after t minutes, T_1 contains $10+3t-t+2t=10+4t$ gal of solution while T_2 contains $10+4t+t-2t=10+3t$ gal of solution. In the time interval $[t, t+\Delta t]$, we have:

$$\begin{aligned}
\Delta y_1 &\approx 3\cdot 3\cdot \Delta t - 1\cdot \frac{y_1}{10+4t}\cdot \Delta t + 2\cdot \frac{y_2}{10+3t}\cdot \Delta t \quad &\Longrightarrow \quad y_1' &= 9 - \frac{1}{10+4t}y_1 + \frac{2}{10+3t}y_2 \\
\Delta y_2 &\approx 4\cdot 2\cdot \Delta t + 1\cdot \frac{y_1}{10+4t}\cdot \Delta t - 2\cdot \frac{y_2}{10+3t}\cdot \Delta t \quad &\Longrightarrow \quad y_2' &= 8 + \frac{1}{10+4t}y_1 - \frac{2}{10+3t}y_2
\end{aligned}$$

We also have the initial conditions $y_1(0)=0,\ y_2(0)=0$.

Optional: in matrix form, writing $\boldsymbol{y}=(y_1, y_2)$, this takes the form

$$\boldsymbol{y}' = \begin{bmatrix} -\frac{1}{10+4t} & \frac{2}{10+3t} \\ \frac{1}{104t} & -\frac{2}{10+3t} \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 9 \\ 8 \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Problem 6. (4 points) Assume that the angle $\theta(t)$ of a swinging pendulum is described by $\theta''+9\theta=0$. Suppose $\theta(0)=2$, $\theta'(0)=-3$. What is the period and the amplitude of the resulting oscillations?

Solution. The characteristic equation has roots $\pm 3i$. Hence, the general solution to the DE is $\theta(t)=A\cos(3t)+B\sin(3t)$.

We use the initial conditions to determine A and B: $\theta(0)=A\overset{!}{=}2$. $\theta'(0)=3B\overset{!}{=}-3$.

Hence, the unique solution to the IVP is $\theta(t)=2\cos(3t)-\sin(3t)$.

In particular, the period is $2\pi/3$ and the amplitude is $\sqrt{A^2+B^2}=\sqrt{2^2+(-1)^2}=\sqrt{5}$.

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(extra scratch paper)