Midterm #2 – Practice

MATH 238 — Differential Equations I
Midterm: Wednesday, Nov 15, 2023

Please print your name:

Reminder. No notes, calculators or tools of any kind will be permitted on the midterm exam.

Problem 1. Let L be a linear differential operator of order 4 with constant real coefficients. Suppose that $3+7i$ is a repeated characteristic root of L.

(a) What is the general solution to $Ly=0$?

(b) Write down the simplest form of a particular solution y_p of the DE $Ly=7x^2e^{3x}$ with undetermined coefficients.

(c) Write down the simplest form of a particular solution y_p of the DE $Ly=e^{3x}\sin(7x)+3x^2$ with undetermined coefficients.

Solution.

(a) Since L is real, if $3+7i$ is a repeated characteristic root of L, then $3-7i$ must be a repeated characteristic root of L as well. Hence, the 4 characteric roots must be $3\pm7i, 3\pm7i$.

The corresponding general solution is $(C_1+C_2x)e^{3x}\cos(7x)+(C_3+C_4x)e^{3x}\sin(7x)$.

(b) The "old" roots are $3\pm7i, 3\pm7i$ while the "new" roots are $3,3,3$.

Hence, there must a particular solution of the form $y_p=(C_1+C_2x+C_3x^2)\,e^{3x}$.

The unique values of C_1,C_2,C_3 for which this is a solution of the DE need to be determined by plugging into the DE.

(c) The "old" roots are $3\pm7i, 3\pm7i$ while the "new" roots are $3\pm7i, 0, 0, 0$.

Hence, there must a particular solution of the form $C_1x^2e^{3x}\cos(7x)+C_2x^2e^{3x}\sin(7x)+C_3+C_4x+C_5x^2$.

The unique values of C_1,C_2,C_3,C_4,C_5 for which this is a solution of the DE need to be determined by plugging into the DE.

Problem 2. Consider a homogeneous linear differential equation with constant real coefficients which has order 8.

(a) Suppose $y(x)=7x-2x^2e^{3x}\sin(5x)$ is a solution. Write down the general solution.

(b) Suppose $y(x)=2xe^{3x}+x\cos(5x)-5\sin(x)$ is a solution. Write down the general solution.

Solution.

(a) The characteristic roots must include $0,0,3\pm5i,3\pm5i,3\pm5i$. Since these are 8 roots and the DE has order 8, there cannot be any additional roots.

Hence, the general solution is $C_1+C_2x+(C_3+C_4x+C_5x^2)e^{3x}\cos(5x)+(C_6+C_7x+C_8x^2)e^{3x}\sin(5x)$.

(b) The characteristic roots must include $3,3,\pm5i,\pm5i,\pm i$. Since these are 8 roots and the DE has order 8, there cannot be any additional roots.

Hence, the general solution is $(C_1+C_2x)e^{3x}+(C_3+C_4x)\cos(5x)+(C_5+C_6x)\sin(5x)+C_7\cos(x)+C_8\sin(x)$.

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Problem 3.

(a) Determine the general solution of the system $\begin{array}{rcl} y_1' & = & y_1 - 6y_2 \\ y_2' & = & y_1 - 4y_2 \end{array}$.

(b) Solve the IVP $\begin{array}{rcl} y_1' & = & y_1 - 6y_2 \\ y_2' & = & y_1 - 4y_2 \end{array}$ with $\begin{array}{rcl} y_1(0) & = & 4 \\ y_2(0) & = & 1 \end{array}$.

(c) Determine a particular solution to $\begin{array}{rcl} y_1' & = & y_1 - 6y_2 \\ y_2' & = & y_1 - 4y_2 - 2e^{3x} \end{array}$.

(d) Determine the general solution to $\begin{array}{rcl} y_1' & = & y_1 - 6y_2 \\ y_2' & = & y_1 - 4y_2 - 2e^{3x} \end{array}$.

Solution.

(a) Using $y_2 = \frac{1}{6}(y_1 - y_1')$ (from the first equation) in the second equation, we get $\frac{1}{6}(y_1' - y_1'') = y_1 - \frac{4}{6}(y_1 - y_1')$.

Simplified (and both sides multiplied by -6), this is $y_1'' + 3y_1' + 2y_1 = 0$.

This is a homogeneous linear DE with constant coefficients. The characteristic roots are $-1, -2$.

Hence, $y_1 = C_1 e^{-x} + C_2 e^{-2x}$.

We can then determine y_2 as $y_2 = \frac{1}{6}(y_1 - y_1') = \frac{1}{6}(C_1 e^{-x} + C_2 e^{-2x} - (-C_1 e^{-x} - 2C_2 e^{-2x})) = \frac{1}{3}C_1 e^{-x} + \frac{1}{2}C_2 e^{-2x}$.

(b) From the previous part, we know $y_1 = C_1 e^{-x} + C_2 e^{-2x}$ and $y_2 = \frac{1}{3}C_1 e^{-x} + \frac{1}{2}C_2 e^{-2x}$.

We solve for C_1 and C_2 using the initial conditions: $y_1(0) = C_1 + C_2 \overset{!}{=} 4$ and $y_2(0) = \frac{1}{3}C_1 + \frac{1}{2}C_2 \overset{!}{=} 1$.

Solving these two equations, we find $C_1 = 6$ and $C_2 = -2$.

Thus, the unique solution to the IVP is $y_1 = 6e^{-x} - 2e^{-2x}$ and $y_2 = 2e^{-x} - e^{-2x}$.

(c) We proceed as in the first part to write $y_2 = \frac{1}{6}(y_1 - y_1')$.

Using this in the second equation and simplifying, we get $y_1'' + 3y_1' + 2y_1 = 12e^{3x}$.

This is an inhomogeneous linear DE with constant coefficients. Since the "old" roots are $-1, -2$, while the "new" root is 3, there must a particular solution of the form $y_1 = Ce^{3x}$ with undetermined coefficient C. To determine C, we plug this y_1 into the DE: $y_1'' + 3y_1' + 2y_1 = (9 + 3 \cdot 3 + 2)Ce^{3x} = 20Ce^{3x} \overset{!}{=} 12e^{3x}$. Hence, $C = \frac{3}{5}$.

Having found $y_1 = \frac{3}{5}e^{3x}$, we can then determine y_2 as $y_2 = \frac{1}{6}(y_1 - y_1') = \frac{1}{6}\left(\frac{3}{5}e^{3x} - \frac{9}{5}e^{3x}\right) = -\frac{1}{5}e^{3x}$.

(d) We get the general solution by adding the particular solution (previous part) and the general solution to the corresponding homogeneous equation (first part):

Hence, the general solution is $y_1 = \frac{3}{5}e^{3x} + C_1 e^{-x} + C_2 e^{-2x}$ and $y_2 = -\frac{1}{5}e^{3x} + \frac{1}{3}C_1 e^{-x} + \frac{1}{2}C_2 e^{-2x}$.

Alternatively. Here is a solution that proceeds from scratch (rather than referring to previous parts):

Using $y_2 = \frac{1}{6}(y_1 - y_1')$ (from the first equation) in the second equation, we get $\frac{1}{6}(y_1' - y_1'') = y_1 - \frac{4}{6}(y_1 - y_1') - 2e^{3x}$.

Simplified (and both sides multiplied by -6), this is $y_1'' + 3y_1' + 2y_1 = 12e^{3x}$.

This is an inhomogeneous linear DE with constant coefficients. Since the "old" roots are $-1, -2$, while the "new" root is 3, there must a particular solution of the form $y_1 = Ce^{3x}$ with undetermined coefficient C. To determine C, we plug this y_1 into the DE: $y_1'' + 3y_1' + 2y_1 = (9 + 3 \cdot 3 + 2)Ce^{3x} = 20Ce^{3x} \overset{!}{=} 12e^{3x}$. Hence, $C = \frac{3}{5}$ and the particular solution is $y_1 = \frac{3}{5}e^{3x}$. The corresponding general solution is $y_1 = \frac{3}{5}e^{3x} + C_1 e^{-x} + C_2 e^{-2x}$.

We can then determine y_2 as follows:

$y_2 = \frac{1}{6}(y_1 - y_1') = \frac{1}{6}\left(\frac{3}{5}e^{3x} + C_1 e^{-x} + C_2 e^{-2x} - \left(\frac{9}{5}e^{3x} - C_1 e^{-x} - 2C_2 e^{-2x}\right)\right) = -\frac{1}{5}e^{3x} + \frac{1}{3}C_1 e^{-x} + \frac{1}{2}C_2 e^{-2x}$.

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Problem 4.

(a) Write the (third-order) differential equation $y''' + 2y'' - 4y' + 5y = 2\sin(x)$ as a system of (first-order) differential equations.

(b) Consider the following system of (second-order) initial value problems:

$$\begin{aligned} y_1'' &= 5y_1' + 2y_2' + e^{2x} \\ y_2'' &= 7y_1 - 3y_2 - 3e^x \end{aligned} \qquad y_1(0) = 1,\ y_1'(0) = 4,\ y_2(0) = 0,\ y_2'(0) = -1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y},\ \boldsymbol{y}(0) = \boldsymbol{c}$.

Solution.

(a) Write $y_1 = y,\ y_2 = y'$ and $y_3 = y''$.

Then, the DE translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -5y_1 + 4y_2 - 2y_3 + 2\sin(x) \end{cases}$.

In matrix form, with $\boldsymbol{y} = (y_1, y_2, y_3)$, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & 4 & -2 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 0 \\ 0 \\ 2\sin(x) \end{bmatrix}$.

(b) Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, with $\boldsymbol{y} = (y_1, y_2, y_3, y_4)$, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 2 \\ 7 & -3 & 0 & 0 \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 0 \\ 0 \\ e^{2x} \\ -3e^x \end{bmatrix}, \qquad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix}.$$

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Problem 5. The mixtures in three tanks T_1, T_2, T_3 are kept uniform by stirring. Brine containing 2 lb of salt per gallon enters the first tank at a rate of 15 gal/min. Mixed solution from T_1 is pumped into T_2 at a rate of 10 gal/min and from T_2 into T_3 at a rate of 10 gal/min. Each tank initially contains 10 gal of pure water. Denote by $y_i(t)$ the amount (in pounds) of salt in tank T_i at time t (in minutes). Derive a system of linear differential equations for the y_i, including initial conditions.

Solution. Note that at time t, T_1 contains $10 + 15t - 10t = 10 + 5t$ gal of solution. On the other hand, T_2 contains a constant amount of 10 gal, and T_3 $10 + 10t$ gal of solution.

In the time interval $[t, t + \Delta t]$, we have:

$$\begin{aligned} \Delta y_1 &\approx 15 \cdot 2 \cdot \Delta t - 10 \cdot \frac{y_1}{10 + 5t} \cdot \Delta t & &\Longrightarrow & y_1' &= 30 - \frac{2y_1}{2 + t} \\ \Delta y_2 &\approx 10 \cdot \frac{y_1}{10 + 5t} \cdot \Delta t - 10 \cdot \frac{y_2}{10} \cdot \Delta t & &\Longrightarrow & y_2' &= \frac{2y_1}{2 + t} - y_2 \\ \Delta y_3 &\approx 10 \cdot \frac{y_2}{10} \cdot \Delta t & &\Longrightarrow & y_3' &= y_2 \end{aligned}$$

We also have the initial conditions $y_1(0) = 0$, $y_2(0) = 0$, $y_3(0) = 0$. In matrix form, writing $\boldsymbol{y} = (y_1, y_2, y_3)$, this is

$$\boldsymbol{y}' = \begin{bmatrix} -\frac{2}{2+t} & 0 & 0 \\ \frac{2}{2+t} & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 30 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is a system of linear inhomogeneous differential equations with non-constant coefficients.

Comment. Because of its particularly simple structure, we actually have the techniques to solve this system. Namely, note that the first equation only involves y_1. It is a linear first-order equation which we could therefore solve using an integrating factor. With y_1 determined, the second differential equation only involves y_2 and is, again, a linear first-order equation. Solving it for y_2, we then get y_3 by a final integration.

Problem 6.

(a) What is the period and the amplitude of $3\cos(7t) - 5\sin(7t)$?

(b) Assume that the angle $\theta(t)$ of a swinging pendulum is described by $\theta'' + 4\theta = 0$. Suppose $\theta(0) = \frac{3}{10}$ and $\theta'(0) = -\frac{4}{5}$. What is the period and the amplitude of the resulting oscillations?

(c) The position $y(t)$ of a certain mass on a spring is described by $y'' + dy' + 5y = 0$. For which value of $d > 0$ is the system underdamped? Critically damped? Overdamped?

(d) A forced mechanical oscillator is described by $y'' + 2y' + y = 25\cos(2t)$. As $t \to \infty$, what is the period and the amplitude of the resulting oscillations?

(e) The motion of a certain mass on a spring is described by $y'' + y' + \frac{1}{2}y = 5\sin(t)$ with $y(0) = 2$ and $y'(0) = 0$. Determine $y(t)$. As $t \to \infty$, what are the period and amplitude of the oscillations?

Solution.

(a) The period is $2\pi/7$ and the amplitude is $\sqrt{3^2 + (-5)^2} = \sqrt{34}$.

(b) The characteristic equation has roots $\pm 2i$. Hence, the general solution to the DE is $\theta(t) = A\cos(2t) + B\sin(2t)$.

We use the initial conditions to determine A and B: $\theta(0) = A = \frac{3}{10}$. $\theta'(0) = 2B = -\frac{4}{5}$.

Hence, the unique solution to the IVP is $\theta(t) = \frac{3}{10}\cos(2t) - \frac{2}{5}\sin(2t)$.

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In particular, the period is π and the amplitude is $\sqrt{A^2+B^2} = \sqrt{\frac{9}{100}+\frac{16}{100}} = \frac{1}{2}$.

(c) The characteristic equation has roots $\frac{1}{2}\left(-d \pm \sqrt{d^2-20}\right)$. The system is underdamped if the solutions involve oscillations, which happens if and only if d^2-20 (the discriminant) is negative.

Since $d^2-20<0$ if $d<\sqrt{20}$, the system is underdamped for $d \in (0, \sqrt{20})$.

Correspondingly, the system is critically damped for $d=\sqrt{20}$ and overdamped for $d>\sqrt{20}$.

(d) The "old" roots are $-1, -1$ while the "new" roots are $\pm 2i$. Since they don't overlap, there must be a particular solution y_p of the form $y_p = A\cos(2t) + B\sin(2t)$.

We plug into the DE to find $y_p'' + 2y_p' + y_p = (-4A + 4B + A)\cos(2t) + (-4B - 4A + B)\sin(2t) \overset{!}{=} 25\cos(2t)$.

Comparing coefficients, we get $-3A + 4B = 25$ and $-3B - 4A = 0$. Solving these, we find $A=-3$ and $B=4$.

Hence, $y_p(t) = -3\cos(2t) + 4\sin(2t)$ and the general solution is $y(t) = -3\cos(2t) + 4\sin(2t) + (C_1 + C_2 x)e^{-t}$.

As $t \to \infty$, we have $e^{-t} \to \infty$ so that $y(t) \approx -3\cos(2t) + 4\sin(2t)$.

In particular, the period is π and the amplitude is $\sqrt{(-3)^2+4^2} = 5$.

(e) The "old" roots are $\frac{-2 \pm \sqrt{4-8}}{4} = -\frac{1}{2} \pm \frac{1}{2}i$ while the "new" roots are $\pm i$. Since there is no overlap, there must be a particular solution y_p of form $y_p = A\cos(t) + B\sin(t)$. By plugging into DE, we find $A=-4$, $B=-2$.

Hence, the general solution is $y(t) = -4\cos(t) - 2\sin(t) + e^{-t/2}(C_1\cos(t/2) + C_2\sin(t/2))$.

We determine C_1 and C_2 using the initial conditions. From $y(0) = -4 + C_1 \overset{!}{=} 2$, we conclude $C_1 = 6$. We then compute $y'(t) = 4\sin(t) - 2\cos(t) - \frac{1}{2}e^{-t/2}(C_1\cos(t/2) + C_2\sin(t/2)) + e^{-t/2}\left(-\frac{1}{2}C_1\sin(t/2) + \frac{1}{2}C_2\cos(t/2)\right)$. Hence, $y'(0) = -2 - \frac{1}{2}C_1 + \frac{1}{2}C_2 \overset{!}{=} 0$, from which we conclude that $C_2 = 10$.

Therefore, the unique solution to the IVP is $y(t) = -4\cos(t) - 2\sin(t) + e^{-t/2}(6\cos(t/2) + 10\sin(t/2))$.

For large t, $y(t) \approx -4\cos(t) - 2\sin(t)$ (since $e^{-t/2} \to 0$). Hence, the period is 2π and the amplitude is $\sqrt{4^2+2^2} = \sqrt{20}$.

Problem 7. The position $y(t)$ of a certain mass on a spring is described by $2y'' + dy' + 3y = F\sin(4\omega t)$.

(a) Assume first that there is no external force, i.e. $F=0$. For which values of d is the system overdamped?

(b) Now, $F \neq 0$ and the system is undamped, i.e. $d=0$. For which values of ω, if any, does resonance occur?

Solution.

(a) The discriminant of the characteristic equation is d^2-24. Hence the system is overdamped if $d^2-24>0$, that is $d>\sqrt{24} = 2\sqrt{6}$.

(b) The natural frequency is $\sqrt{\frac{3}{2}}$. Resonance therefore occurs if $4\omega = \sqrt{\frac{3}{2}}$ or, equivalently, $\omega = \frac{1}{4}\sqrt{\frac{3}{2}}$.

Problem 8.

(a) Determine the general solution to $y'' - 4y' + 4y = 3e^{2x}$.

(b) Determine the general solution to the differential equation $y''' - y = e^x + 7$.

(c) Determine the general solution $y(x)$ to the differential equation $y^{(4)} + 6y''' + 13y'' = 2$. Express the solution using real numbers only.

(d) Solve the initial value problem $y''+2y'+y=2e^{2x}+e^{-x}$, $y(0)=-1$, $y'(0)=2$.

Solution.

(a) The characteristic equation for the corresponding homogeneous DE has roots $2,2$ ("old" roots). The right-hand side solves a DE whose characteristic equation has root 2 ("new" root). Hence, by the method of undetermined coefficients, there must be a particular solution of the form $y_p=Ax^2e^{2x}$.

To determine A, we plug into the DE using $y_p'=2A(x+x^2)e^{2x}$ and $y_p''=2A(1+4x+2x^2)e^{2x}$:
$y_p''-4y_p'+4y_p=[2A(1+4x+2x^2)-8A(x+x^2)+4Ax^2]e^{2x}=2Ae^{2x}\overset{!}{=}3e^{2x}$. Hence, $A=\frac{3}{2}$ so that $y_p=\frac{3}{2}x^2e^{2x}$.

Accordingly, the general solution is $y(x)=\left(C_1+C_2x+\frac{3}{2}x^2\right)e^{2x}$.

(b) Let us first solve the homogeneous equation $y'''-y=0$. Its characteristic polynomial $D^3-1=(D-1)(D^2+D+1)$ has roots 1 and $-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$ ("old" roots). The right-hand side solves a DE whose characteristic equation has roots $0,1$ ("new" roots).

Noting the repetition of the root 1, by the method of undetermined coefficients, there must be a particular solution of the form $y_p=Axe^x+B$.

$y_p'=A(x+1)e^x$, $y_p''=A(x+2)e^x$, $y_p'''=A(x+3)e^x$

Plugging into the DE, we get $y_p'''-y_p=3Ae^x-B\overset{!}{=}e^x+7$. Consequently, $A=\frac{1}{3}$, $B=-7$ so that $y_p=-7+\frac{1}{3}xe^x$.

Hence, the general solution is $y(x)=-7+(C_1+\frac{1}{3}x)e^x+C_2e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)+C_3e^{-x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$.

(c) Since $D^4+6D^3+13D^2=D^2(D^2+6D+13)$, the characteristic equation for the corresponding homogeneous DE has roots $0,0,-3\pm2i$ ("old" roots). The right-hand side solves a DE whose characteristic equation has root 0 ("new" root). Hence, by the method of undetermined coefficients, there must be a particular solution of the form $y_p=Ax^2$.

Plugging into the DE, we get $y_p^{(4)}+6y_p'''+13y_p''=26A\overset{!}{=}2$. Thus $A=\frac{1}{13}$ so that $y_p=\frac{1}{13}x^2$.

Hence, the general solution is $y(x)=\frac{1}{13}x^2+C_1+C_2x+C_3e^{-3x}\cos(2x)+C_4e^{-3x}\sin(2x)$.

(d) The characteristic equation for the associated homogeneous DE has roots $-1,-1$ (the "old" roots). The right-hand side solves a DE whose characteristic equation has roots $-1,2$ (the "new" roots).

Hence, by the method of undetermined coefficients, there must be a particular solution of the form $y_p=Ae^{2x}+Bx^2e^{-x}$. To find A,B we plug into the DE. [...] We find $A=\frac{2}{9}$ and $B=\frac{1}{2}$.

Particular solution: $y_p=\frac{2}{9}e^{2x}+\frac{1}{2}x^2e^{-x}$

General solution: $y=\frac{2}{9}e^{2x}+\frac{1}{2}x^2e^{-x}+C_1e^{-x}+C_2xe^{-x}$

Now, we use the initial values [...], to find $y(x)=\frac{2}{9}e^{2x}+\frac{1}{2}x^2e^{-x}-\frac{11}{9}e^{-x}+\frac{1}{3}xe^{-x}$.

Problem 9.

(a) Consider the differential equation $x^2y''-4xy'+6y=0$. Find all solutions of the form $y(x)=x^r$.

(b) Determine the general solution of $x^2y''-4xy'+6y=x^3$.

Solution.

(a) Plugging $y(x)=x^r$ into the DE, we get $x^2r(r-1)x^{r-2}-4xrx^{r-1}+6x^r=[r(r-1)-4r+6]x^r\overset{!}{=}0$.

Since $r(r-1)-4r+6=(r-2)(r-3)$, we find the solutions x^2 and x^3. Since this is a second-order equation and our solutions are independent, there can be no further solutions.

(b) We can find a particular solution to this inhomogeneous DE using the method of variation of parameters/constants. From the first part, we know that the corresponding homogeneous DE has the solutions $y_1 = x^2$, $y_2 = x^3$. The Wronskian of these is $W(x) = y_1 y_2' - y_1' y_2 = x^4$.

Put the DE in the form $y'' - 4x^{-1} y' + 6x^{-2} y = f(x)$ with $f(x) = x$. Then, by the method of variation of parameters, a particular solution is given by

$$y_p = -y_1(x) \int \frac{y_2(x) f(x)}{W(x)} \mathrm{d}x + y_2(x) \int \frac{y_1(x) f(x)}{W(x)} \mathrm{d}x = -x^2 \int 1 \, \mathrm{d}x + x^3 \int \frac{1}{x} \mathrm{d}x = -x^3 + x^3 \ln|x|.$$

Hence, the general solution is $y(x) = C_1 x^2 + (C_2 + \ln|x|) x^3$.

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