Midterm #1 – Practice

MATH 238 — Differential Equations I
Midterm: Wednesday, Oct 4, 2023

Please print your name:

Bonus challenge. Let me know about any typos you spot in the posted solutions (or lecture sketches). Any mathematical typo, that is not yet fixed by the time you send it to me, is worth a bonus point.

Reminder. No notes, calculators or tools of any kind will be permitted on the midterm exam.

Problem 1. Find the general solution to $y'' + y' = 12y$.

Solution. We look for solutions of the form e^{rx}.

Plugging e^{rx} into the DE, we get $r^2 e^{rx} + r e^{rx} = 12 e^{rx}$ which simplifies to $r^2 + r - 12 = 0$.

This quadratic equation has the solutions $r = \frac{-1 \pm \sqrt{1 - 4 \cdot (-12)}}{2} = \frac{-1 \pm 7}{2} = -4, 3$.

This means we found the two solutions $y_1 = e^{-4x}$, $y_2 = e^{3x}$.

The general solution to the DE is $C_1 e^{-4x} + C_2 e^{3x}$.

Problem 2. Consider the initial value problem

$$(xy + 2x) y' = \cos(x), \qquad y(a) = b.$$

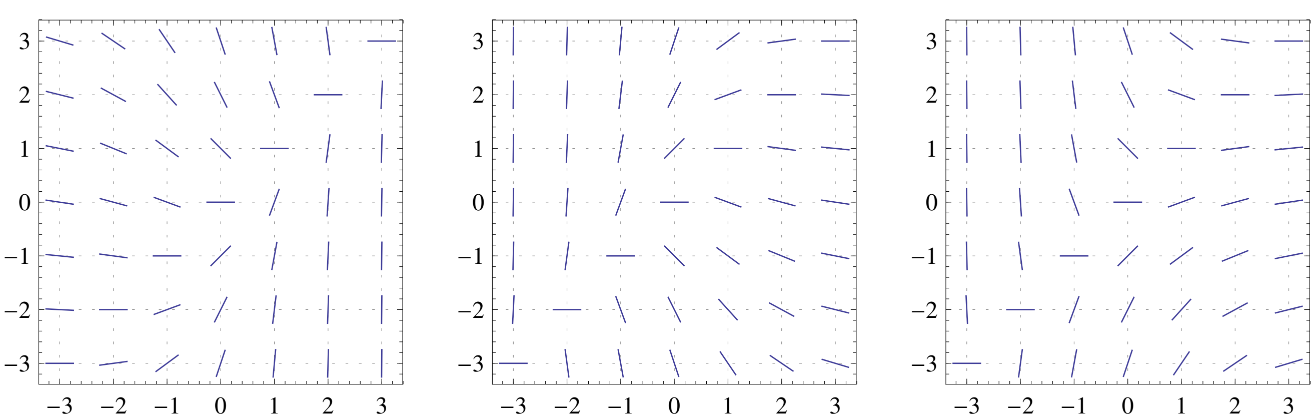
For which values of a and b can we guarantee existence and uniqueness of a (local) solution?

Solution. Let us write $y' = f(x, y)$ with $f(x, y) = \frac{\cos(x)}{xy + 2x} = \frac{\cos(x)}{x(y+2)}$. Then $\frac{\partial}{\partial y} f(x, y) = -\frac{\cos(x)}{x(y+2)^2}$.

Both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $x \neq 0$ and $y \neq -2$.

Hence, if $a \neq 0$ and $b \neq -2$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Problem 3. Circle the slope field below which belongs to the differential equation $e^x y' = x - y$.



Solution. A good point to carefully consider is $(1, 3)$. By the DE, a solution passing through that point has slope y' satisfying $e^1 y' = 1 - 3$. Equivalently, $y' = -2/e$. The only plot compatible with that is the third one.

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Of course, we can arrive at the same conclusion based on other points or, even better, based on several points.

Problem 4. Consider the IVP $(2x+1)\frac{\mathrm{d}y}{\mathrm{d}x} = y + 2x + 1,\ y(2) = 0.$

(a) Approximate the solution $y(x)$ for $x \in [2,3]$ using Euler's method with 2 steps. In particular, what is the approximation of $y(3)$?

(b) **(use calculator)** Likewise approximate the solution $y(x)$ for $x \in [2,3]$ using Euler's method with 3 steps.

(c) Solve this IVP exactly. In particular, what is the exact value of $y(3)$? Compare with the approximate values at $x = 3$.

Solution.

(a) First, we need to put the DE into the form $y' = f(x,y)$ by rewriting it as $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{2x+1} + 1$.

The step size is $h = \frac{3-2}{2} = \frac{1}{2}$. We apply Euler's method with $f(x,y) = \frac{y}{2x+1} + 1$:

$$\begin{array}{ll} x_0 = 2 & y_0 = 0 \\ x_1 = \frac{5}{2} & y_1 = y_0 + h f(x_0, y_0) = 0 + \frac{1}{2} \cdot \left[\frac{0}{2 \cdot 2 + 1} + 1\right] = \frac{1}{2} \\ x_2 = 3 & y_2 = y_1 + h f(x_1, y_1) = \frac{1}{2} + \frac{1}{2} \cdot \left[\frac{\frac{1}{2}}{2 \cdot \frac{5}{2} + 1} + 1\right] = \frac{25}{24} \end{array}$$

In particular, the approximation for $y(3)$ is $y_2 = \frac{25}{24} \approx 1.042$.

(b) The step size is $h = \frac{3-2}{3} = \frac{1}{3}$. We again apply Euler's method with $f(x,y) = \frac{y}{2x+1} + 1$:

$$\begin{array}{ll} x_0 = 2 & y_0 = 0 \\ x_1 = \frac{7}{3} & y_1 = y_0 + h f(x_0, y_0) = 0 + \frac{1}{3} \cdot \left[\frac{0}{2 \cdot 2 + 1} + 1\right] = \frac{1}{3} \\ x_2 = \frac{8}{3} & y_2 = y_1 + h f(x_1, y_1) = \frac{1}{3} + \frac{1}{3} \cdot \left[\frac{\frac{1}{3}}{2 \cdot \frac{7}{3} + 1} + 1\right] = \frac{35}{51} \\ x_3 = 3 & y_3 = y_2 + h f(x_2, y_2) = \frac{35}{51} + \frac{1}{3} \cdot \left[\frac{\frac{35}{51}}{2 \cdot \frac{8}{3} + 1} + 1\right] = \frac{341}{323} \end{array}$$

In particular, the approximation for $y(3)$ is $y_3 = \frac{341}{323} \approx 1.056$.

(c) This is a linear DE.

- We rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{2x+1} y = 1$ (standard form).
- The integrating factor is $f(x) = \exp\left(\int -\frac{1}{2x+1}\,\mathrm{d}x\right) = \exp\left(-\frac{1}{2}\ln(2x+1)\right) = (2x+1)^{-1/2}$.
- Multiply the (rewritten) DE by $f(x)$ to get $\underbrace{(2x+1)^{-1/2}\frac{\mathrm{d}y}{\mathrm{d}x} - (2x+1)^{-3/2} y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[(2x+1)^{-1/2} y]} = (2x+1)^{-1/2}$.
- Integrate both sides to get $(2x+1)^{-1/2} y = \int (2x+1)^{-1/2}\mathrm{d}x = (2x+1)^{1/2} + C$.
- Substituting $x = 2$ and $y = 0$ (because $y(2) = 0$), we get $0 = \sqrt{5} + C$ so that $C = -\sqrt{5}$.

Hence, the (unique) exact solution of the IVP is $y(x) = (2x+1)^{1/2}((2x+1)^{1/2} - \sqrt{5}) = 2x + 1 - \sqrt{5(2x+1)}$.

In particular, the exact value at $x = 3$ is $y(3) = 7 - \sqrt{35} \approx 1.084$.

We observe that our approximations for $y(3) \approx 1.084$ improved from 1.042 to 1.056 as we increased the number of steps (equivalently, we decreased the step size h from $\frac{1}{2}$ to $\frac{1}{3}$).

Problem 5. In the differential equation $x(y+1)\frac{\mathrm{d}y}{\mathrm{d}x} = (x^2+y)^3$ substitute $u = (x^2+y)^3$.

What is the resulting differential equation for u? No need to simplify!

Solution. If $u = (x^2+y)^3$, then $y = u^{1/3} - x^2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}u^{-2/3}\frac{\mathrm{d}u}{\mathrm{d}x} - 2x$.

Hence, the resulting differential equation for u is $x(u^{1/3} - x^2 + 1)\left(\frac{1}{3}u^{-2/3}\frac{\mathrm{d}u}{\mathrm{d}x} - 2x\right) = u$.

Problem 6. Solve the initial value problem $y' = 2xy + 3x^2 e^{x^2}$, $y(0) = 5$.

Solution. This is a linear DE. To solve it, we first bring it in the form $y' - 2xy = 3x^2 e^{x^2}$.

The integrating factor is $\exp(\int -2x\,\mathrm{d}x) = e^{-x^2}$.

We multiply the (rewritten) DE by e^{-x^2} to get $\underbrace{e^{-x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + -2xe^{-x^2}y}_{=\frac{\mathrm{d}}{\mathrm{d}x}[e^{-x^2}y]} = 3x^2$.

We then integrate both sides to get $e^{-x^2}y = x^3 + C$.

Using $y(0) = 5$, we find $5 = C$. Hence the solution is $y(x) = (x^3+5)e^{x^2}$.

Problem 7. Find a general solution to the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + y^2\sin(x) = 0$.

Solution. This DE is separable:

$$\frac{\mathrm{d}y}{y^2} = -\sin(x)\,\mathrm{d}x \quad \Longrightarrow \quad \frac{-1}{y} = \cos(x) + C \quad \Longrightarrow \quad y = \frac{-1}{\cos(x) + C}.$$

In addition, there is the singular solution $y = 0$, which we lost when dividing by y^2.

Problem 8. Find a general solution to the differential equation $xy' = y + x^2\cos(x)$.

Solution. This is a linear DE. To solve it, we first bring it in the form $y' - \frac{1}{x}y = x\cos(x)$.

The integrating factor is $e^{\int -\frac{1}{x}\mathrm{d}x} = e^{-\ln x} = \frac{1}{x}$.

We multiply the (rewritten) DE by $\frac{1}{x}$ to get $\underbrace{\frac{1}{x}y' - \frac{1}{x^2}y}_{=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x}y\right]} = \cos(x)$.

We then integrate both sides to get $\frac{y}{x} = \sin(x) + C$.

Hence a general solution is $y = x\sin(x) + Cx$.

Problem 9. A tank contains 20gal of pure water. It is filled with brine (containing 5lb/gal salt) at a rate of 8gal/min. At the same time, well-mixed solution flows out at a rate of 6gal/min. How much salt is in the tank after t minutes?

Solution. Let $x(t)$ denote the amount of salt (in lb) in the tank after time t (in min).

At time t, the concentration of salt (in lb/gal) in the tank is $\frac{x(t)}{V(t)}$ where $V(t) = 20 + (8-6)t = 20 + 2t = 2(10+t)$ is the volume (in gal) in the tank.

In the time interval $[t, t+\Delta t]$: $\Delta x \approx 8 \cdot 5 \cdot \Delta t - 6 \cdot \frac{x(t)}{V(t)} \cdot \Delta t = 40 \cdot \Delta t - 3 \cdot \frac{x(t)}{10+t} \cdot \Delta t$.

Hence, $x(t)$ solves the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = 40 - \frac{3}{10+t} x$ with $x(0) = 0$. Since this is a linear DE, we can solve it as follows:

- We write it in the form $\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{3}{10+t} x = 40$.
- The integrating factor is $f(t) = \exp\left(\int \frac{3}{10+t} \mathrm{d}t\right) = \exp(3\ln(10+t)) = (10+t)^3$.
- Multiply the (rewritten) DE by $f(t) = (10+t)^3$ to get $\underbrace{(10+t)^3 \frac{\mathrm{d}x}{\mathrm{d}t} + 3(10+t)^2 x}_{= \frac{\mathrm{d}}{\mathrm{d}t}[(10+t)^3 x]} = 40(10+t)^3$.
- Integrate both sides to get $(10+t)^3 x = 40 \int (10+t)^3 \mathrm{d}t = 10(10+t)^4 + C$.

Hence the general solution to the DE is $x(t) = 10(10+t) + \frac{C}{(10+t)^3}$. Using $x(0) = 0$, we find $100 + \frac{C}{10^3} = 0$ from which we conclude that $C = -100 \cdot 10^3 = -100{,}000$.

After t minutes, the tank therefore contains $x(t) = 10(10+t) - \frac{100{,}000}{(10+t)^3}$ pounds of salt.

Comment. If we don't simplify $\frac{6}{20+2t} = \frac{3}{10+t}$, then a possible integrating factor is $f(t) = \exp\left(\int \frac{6}{20+2t} \mathrm{d}t\right) = \exp(3\ln(20+2t)) = (20+2t)^3$. However, note that this is just 2^3 times the factor that we obtained above. (In general, the integrating factor is only unique up to a constant multiple because we can choose any convenient constant of integration when computing it.)

Problem 10. The time rate of change of a rabbit population P is proportional to the square root of P. At time $t = 0$, the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be after two months?

Solution. $P'(t) = k\sqrt{P}$ and $P(0) = 100$, $P'(0) = 20$. The problem asks for $P(2)$.

At $t = 0$, we have $20 = P'(0) = k\sqrt{P(0)} = 10k$. Hence $k = 2$.

The DE $\frac{\mathrm{d}P}{\mathrm{d}t} = 2\sqrt{P}$ is separable: $P^{-1/2}\mathrm{d}P = 2\mathrm{d}t$.

Integrating both sides, we find $2\sqrt{P} = 2t + C$. From $P(0) = 100$, we conclude that $C = 20$.

Thus, $P(t) = (t+10)^2$. In particular, there will be $P(2) = 12^2 = 144$ rabbits after two months.

Problem 11. A rising population is modeled by the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t} = 1000P - 20P^2$.

(a) When the population size stabilizes in the long term, how big will the population be?

(b) Under which condition will the population size shrink?

(c) What is the population size when it is growing the fastest?

Solution.

(a) Once the population reaches a stable level in the long term, we have $\frac{\mathrm{d}P}{\mathrm{d}t} = 0$ (no change in population size).

Hence, $0 = 1000P - 20P^2 = 20P(50 - P)$ which implies that $P = 0$ or $P = 50$. Since the population is rising, it will approach 50 in the long term.

(b) The population size will shrink if $\frac{dP}{dt} < 0$.

The DE tells us that is the case if and only if $1000P - 20P^2 < 0$ or, equivalently, if $P > \frac{1000}{20} = 50$.

(c) This is asking when $\frac{dP}{dt}$ (the population growth) is maximal.

The DE is telling us that this growth is $f(P) = 1000P - 20P^2$. This a parabola that opens to the bottom. From Calculus, we know that it has a global maximum when $f'(P) = 0$.

$f'(P) = 1000 - 40P = 0$ leads to $P = 25$.

Thus, the population is growing the fastest when its size is 25.

Comment. You probably noticed that the DE is the logistic equation. The first problem therefore is asking about the carrying capacity which we could determine by matching parameters. However, note that we are able to (easily!) decide all the above questions directly from the DE without needing the solution of the logistic equation.

Problem 12. Solve the initial value problem $x\frac{dy}{dx} = y - xe^{y/x}, \quad y(1) = 0.$

Solution. This DE is neither separable nor linear. Hence, we look for a suitable substitution.

Since the right-hand side features a y, we divide both sides by x. We get $\frac{dy}{dx} = \frac{y}{x} - e^{y/x}$, which is of the form $y' = F\left(\frac{y}{x}\right)$.

We therefore substitute $u = \frac{y}{x}$. Then $y = ux$ and $\frac{dy}{dx} = x\frac{du}{dx} + u$.

The resulting DE is $x\frac{du}{dx} + u = u - e^u$, which simplifies to $x\frac{du}{dx} = -e^u$.

This DE is separable: $e^{-u}\,du = -\frac{1}{x}\,dx$. Integrating both sides, we find $-e^{-u} = -\ln|x| + C$.

Accordingly, for the initial DE, $-e^{-y/x} = -\ln(x) + C$. (Note that $x > 0$, at least locally, due to the initial condition.)

Using $y(1) = 0$ we find $-e^{0/1} = -\ln(1) + C$ so that $C = -1$.

Thus $e^{-y/x} = \ln(x) + 1$ and, therefore, $y = -x\ln(\ln(x) + 1)$.

Problem 13. Solve the initial value problem $(x^2+1)\frac{dy}{dx} + xy = \frac{1}{\sqrt{x^2+1}}, \quad y(0) = 1.$

Solution. This is a linear DE. To solve it, we first bring it in the form $\frac{dy}{dx} + \frac{x}{x^2+1}y = \frac{1}{(x^2+1)^{3/2}}$.

The integrating factor is $\exp\left(\int \frac{x}{x^2+1}\,dx\right) = \exp\left(\frac{1}{2}\ln(x^2+1)\right) = (x^2+1)^{1/2}$.

We multiply the (rewritten) DE by $(x^2+1)^{1/2}$ to get $\underbrace{(x^2+1)^{1/2}\frac{dy}{dx} + \frac{x}{(x^2+1)^{1/2}}y}_{=\frac{d}{dx}[(x^2+1)^{1/2}y]} = \frac{1}{x^2+1}$.

We then integrate both sides to get $(x^2+1)^{1/2}y = \arctan(x) + C$.

Since $y(0) = 1$, we find $C = 1$. Therefore,

$$y(x) = \frac{\arctan(x) + 1}{(x^2+1)^{1/2}}.$$

Problem 14. Find a general solution to the differential equation $2 + \frac{dy}{dx} = \sqrt{2x+y}$.

Solution. This DE is neither separable nor linear. Hence, we look for a suitable substitution.

Note that this DE is of the form $y' = F(2x+y)$ with $F(t) = \sqrt{t} - 2$.

We therefore substitute $u = 2x + y$. Then $y = u - 2x$ and $\frac{dy}{dx} = \frac{du}{dx} - 2$.

The resulting DE is $2 + \left(\frac{du}{dx} - 2\right) = \sqrt{u}$ or, simplified, $\frac{du}{dx} = \sqrt{u}$.

This DE is separable: $u^{-1/2}\,du = dx$. Integrating both sides, we find $2u^{1/2} = x + C$.

Hence $u = \frac{1}{4}(x+C)^2$ and $y = u - 2x = \frac{1}{4}(x+C)^2 - 2x$ [which is a solution as long as $x + C > 0$].

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