The fin equation from thermodynamics

The following is an example from thermodynamics. The governing differential equation is a second-order DE that is like the equation describing the motion of a mass on a spring ($my''+ky=0$) except that one term has the opposite sign. Besides showcasing an application, we want to show off how $\cosh$ and $\sinh$ are useful for writing certain solutions in a more pleasing form.

Let $T(x)$ describe the temperature at position x in a fin with fin base at $x=0$ and fin tip at $x=L$.

For more context on fins: https://en.wikipedia.org/wiki/Fin_(extended_surface)

If we write $\theta(x)=T(x)-T_\infty$ for the temperature excess at position x (with T_∞ the external temperature), then we find (under various simplifying assumptions) that the temperature distribution in our fin satisfies the following DE, known as the **fin equation**:

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}-m^2\theta=0, \quad m^2=\frac{hP}{kA}>0.$$

- A is the cross-sectional area of the fin (assumed to be the same for all positions x).
- P is the perimeter of the fin (assumed to be the same for all positions x).
- k is the thermal conductivity of the material (assumed to be constant).
- h is the convection heat transfer coefficient (assumed to be constant).

Since the DE is homogeneous and linear with characteristic roots $\pm m$, the general solution is

$$\theta(x)=C_1e^{mx}+C_2e^{-mx}=D_1\cosh(mx)+D_2\sinh(mx).$$

The constants C_1, C_2 (or, equivalently, D_1, D_2) can then be found by emposing appropriate boundary conditions at the **fin base** ($x=0$) and at the **fin tip** ($x=L$).

In practice, we often know the temperature at the fin base and therefore the temperature excess, resulting in the boundary condition $\theta(0)=\theta_0$. At the fin tip, common boundary conditions are:

- $\theta(L)\to 0$ as $L\to\infty$ (infinitely long fin)

  In this case, the fin is so long that the temperature at the fin tip approaches the external temperature. Mathematically, we get $\theta(x)=Ce^{-mx}$ since $e^{mx}\to\infty$ as $x\to\infty$. It follows from $\theta(0)=\theta_0$ that $C=\theta_0$.

  Thus, the temperature excess is $\theta(x)=\theta_0\, e^{-mx}$.

- $\theta'(L)=0$ (neglible heat loss at the fin tip, "adiabatic fin tip")

  This can be a more reasonable assumption than the infinitely long fin. Note that the total heat transfer from the fin is proportional to its surface area. If the surface area at the fin tip is a negligible fraction of the total surface area, then it is reasonable to assume that $\theta'(L)=0$.

  In this case, the temperature excess is $\theta(x)=\theta_0\frac{\cosh(m(L-x))}{\cosh(mL)}$.

  Check! Instead of computing this from scratch (do that as well, later!), check that this indeed solves the DE as well as the boundary conditions $\theta(0)=\theta_0$ and $\theta'(L)=0$. This should be a rather quick check!

- $\theta(L)=\theta_L$ (specified temperature at fin tip)

  In this case, the temperature excess is $\theta(x)=\frac{\theta_L\sinh(mx)+\theta_0\sinh(m(L-x))}{\sinh(mL)}$.

  Check! Again, check that this indeed solves the DE as well as the boundary conditions $\theta(0)=\theta_0$ and $\theta(L)=\theta_L$. Once more, this should be a quick and pleasant check.

Armin Straub
straub@southalabama.edu

Application to military strategy: Lanchester's equations

In military strategy, Lanchester's equations can be used to model two opposing forces during "aimed fire" battle.

Let $x(t)$ and $y(t)$ describe the number of troops on each side. Then Lanchester (during World War I) assumed that the rates $-x'(t)$ and $-y'(t)$, at which soldiers are put out of action, are proportional to the number of opposing forces. That is:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -\beta y(t) \\ -\alpha x(t) \end{bmatrix}, \quad \text{or, in matrix form: } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -\beta \\ -\alpha & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The proportionality constants $\alpha, \beta > 0$ indicate the strength of the forces ("fighting effectiveness coefficients").

These are simple linear DEs with constant coefficients, which we have learned how to solve.

Comment. The "aimed fire" means that all combatants are engaged, as is common in modern combat with long-range weapons. This is rather different than ancient combat where soldiers were engaging one opponent at a time.

For more details, see: `https://en.wikipedia.org/wiki/Lanchester%27s_laws`

Example 160. Solve Lanchester's equations subject to the initial conditions $x(0) = x_0$ and $y(0) = y_0$.

Solution. (using Laplace transforms) $x' = -\beta y$ transforms into $sX - x_0 = -\beta Y$. Likewise, $y' = -\alpha x$ transforms into $sY - y_0 = -\alpha X$. The transformed equations are regular equations that we can solve for X and Y.

For instance, by the first equation, $Y = -\frac{1}{\beta}(sX - x_0)$.

Used in the second equation, we get $-\frac{s}{\beta}(sX - x_0) - y_0 = -\alpha X$ so that $(s^2 - \alpha\beta)X = sx_0 - \beta y_0$.

Hence, the system is solved by $X = \frac{sx_0 - \beta y_0}{s^2 - \alpha\beta}$ and $Y = -\frac{1}{\beta}(sX - x_0) = \frac{sy_0 - \alpha x_0}{s^2 - \alpha\beta}$.

As a final step, we need to take the inverse Laplace transform to get $x(t) = \mathcal{L}^{-1}(X(s))$ and $y(t) = \mathcal{L}^{-1}(Y(s))$.

Using partial fractions, $X(s) = \frac{sx_0 - \beta y_0}{(s - \sqrt{\alpha\beta})(s + \sqrt{\alpha\beta})} = \frac{A}{s - \sqrt{\alpha\beta}} + \frac{B}{s + \sqrt{\alpha\beta}}$ with

$$A = \left. \frac{sx_0 - \beta y_0}{s + \sqrt{\alpha\beta}} \right|_{s = \sqrt{\alpha\beta}} = \frac{\sqrt{\alpha\beta}\, x_0 - \beta y_0}{2\sqrt{\alpha\beta}} = \frac{1}{2}\left(x_0 - y_0\sqrt{\frac{\beta}{\alpha}} \right), \quad B = \left. \frac{sx_0 - \beta y_0}{s - \sqrt{\alpha\beta}} \right|_{s = -\sqrt{\alpha\beta}} = \frac{1}{2}\left(x_0 + y_0\sqrt{\frac{\beta}{\alpha}} \right).$$

It follows that $x(t) = Ae^{\sqrt{\alpha\beta}t} + Be^{-\sqrt{\alpha\beta}t}$. We obtain a similar formula for $y(t)$ (with x_0 and y_0 as well as α and β swapped for each other).

Solution. (without Laplace transforms) Our goal is to write down a single DE that only involves, say, $x(t)$.

From the first DE, we get $y(t) = -\frac{1}{\beta}x'(t)$. Hence, $y'(t) = -\frac{1}{\beta}x''(t)$. Using that in the second DE, we obtain $-\frac{1}{\beta}x''(t) = -\alpha x(t)$ or, equivalently, $x''(t) - \alpha\beta x(t) = 0$.

Observe that, since $y(t) = -\frac{1}{\beta}x'(t)$, the initial condition $y(0) = y_0$ translates into $x'(0) = -\beta y_0$.

The roots are $\pm r$ where $r = \sqrt{\alpha\beta}$. Hence, $x(t) = C_1 e^{rt} + C_2 e^{-rt}$.

Using the initial conditions $x(0) = x_0$ and $x'(0) = -\beta y_0$, we find $C_1 + C_2 = x_0$ and $rC_1 - rC_2 = -\beta y_0$.

This results in $C_1 = \frac{1}{2}\left(x_0 - \frac{\beta y_0}{r} \right)$ and $C_2 = \frac{1}{2}\left(x_0 + \frac{\beta y_0}{r} \right)$.

Correspondingly, using $r = \sqrt{\alpha\beta}$,

$$x(t) = \frac{1}{2}\left(x_0 - y_0\sqrt{\frac{\beta}{\alpha}} \right) e^{\sqrt{\alpha\beta}t} + \frac{1}{2}\left(x_0 + y_0\sqrt{\frac{\beta}{\alpha}} \right) e^{-\sqrt{\alpha\beta}t}$$

with a similar formula for $y(t) = -\frac{1}{\beta}x'(t)$.

Comment. The formulas take a particularly pleasing form when written in terms of $\cosh$ and $\sinh$ instead:

$$\begin{aligned} x(t) &= x_0\cosh(\sqrt{\alpha\beta}\,t) - y_0\sqrt{\frac{\beta}{\alpha}}\sinh(\sqrt{\alpha\beta}\,t), \\ y(t) &= y_0\cosh(\sqrt{\alpha\beta}\,t) - x_0\sqrt{\frac{\alpha}{\beta}}\sinh(\sqrt{\alpha\beta}\,t). \end{aligned}$$

Example 161. Determine conditions on x_0, y_0 (size of forces) and α, β (effectiveness of forces) that allow us to conclude who will win the battle.

Solution. We analyze our explicit formulas to find out which of $x(t)$ and $y(t)$ becomes 0 first (and therefore loses the battle). Note that both solutions are combinations of $e^{\sqrt{\alpha\beta}\,t}$ and $e^{-\sqrt{\alpha\beta}\,t}$. Further note that the term $e^{\sqrt{\alpha\beta}\,t}$ dominates the other as t gets large.

Since $y(t) = -\frac{1}{\beta}x'(t)$, the coefficients of $e^{\sqrt{\alpha\beta}\,t}$ in the two solutions $x(t)$ and $y(t)$ have opposite signs (for $x(t)$ that coefficient is $\frac{1}{2}(x_0 - y_0\sqrt{\beta/\alpha})$). This allows us to conclude that $x(t)$ wins the battle if $x_0 - y_0\sqrt{\frac{\beta}{\alpha}} > 0$. This is equivalent to $\alpha x_0^2 > \beta y_0^2$.

Solution. (without solving the DE) As an alternative, we can also start fresh and divide the two equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\beta y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -\alpha x$$

to get $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\alpha x}{\beta y}$. Using separation of variables, we find $\beta y\,\mathrm{d}y = \alpha x\,\mathrm{d}x$ which implies $\frac{1}{2}\beta y^2 = \frac{1}{2}\alpha x^2 + D$.

Consequently, $\alpha x^2 - \beta y^2 = C$ where $C = -2D$ is a constant. Using the initial conditions, we find $C = \alpha x_0^2 - \beta y_0^2$. If $y(t)$ is zero first (x wins), then $\alpha x(t)^2 = C > 0$. On the other hand, if $x(t)$ is zero first, then $-\beta y(t)^2 = C < 0$. In other words, the sign of C determines who will win the battle.

Namely, x will win if $C > 0$ which is equivalent to $\alpha x_0^2 > \beta y_0^2$.

Conclusion. The condition we found is known as **Lanchester's square law**: its crucial message is that the sizes x_0, y_0 of the forces count quadratically, whereas the fighting effectivenesses α, β only count linearly. In other words, to beat a force with twice the effectiveness the other side only needs to have a force that is about 41.4% larger (since $\sqrt{2} \approx 1.4142$). Or, put differently, to beat a force of twice the size, the other side would need a fighting effectiveness that is more than 4 times as large.