Further entries in the Laplace transform table

Finally, we expand our table of Laplace transforms to the following:

$f(t)$	$F(s)$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
e^{at}	$\frac{1}{s-a}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{at}f(t)$	$F(s-a)$
$tf(t)$	$-F'(s)$
$u_a(t)f(t-a)$	$e^{-as}F(s)$

Example 152. (new entry) We add the following to our table of Laplace transforms:

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-st}e^{at}f(t)\mathrm{d}t = \int_0^\infty e^{-(s-a)t}f(t)\mathrm{d}t = F(s-a)$$

Example 153. (new entry) We also add the following to our table of Laplace transforms:

$$\mathcal{L}(tf(t)) = \int_0^\infty e^{-st}tf(t)\mathrm{d}t = \int_0^\infty -\frac{\mathrm{d}}{\mathrm{d}s}e^{-st}f(t)\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st}f(t)\mathrm{d}t = -F'(s)$$

In particular,

$$\begin{aligned}
\mathcal{L}(t) &= \mathcal{L}(t\cdot 1) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s} = \frac{1}{s^2}\\
\mathcal{L}(t^2) &= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s^2} = \frac{2}{s^3}\\
&\vdots\\
\mathcal{L}(t^n) &= \frac{n!}{s^{n+1}}.
\end{aligned}$$

Example 154. Determine the Laplace transform $\mathcal{L}((t-3)e^{2t})$.

Solution. $\mathcal{L}((t-3)e^{2t}) = \mathcal{L}(te^{2t}) - 3\mathcal{L}(e^{2t}) = \frac{1}{(s-2)^2} - \frac{3}{s-2}$

Here, we combined $\mathcal{L}(tf(t)) = -F'(s)$ with $\mathcal{L}(e^{2t}) = \frac{1}{s-2}$ to get $\mathcal{L}(te^{2t}) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s-2} = \frac{1}{(s-2)^2}$.

Alternative. Combine $\mathcal{L}(t-3) = \frac{1}{s^2} - \frac{3}{s}$ and $\mathcal{L}(f(t)e^{2t}) = F(s-2)$ to again get $\mathcal{L}((t-3)e^{2t}) = \frac{1}{(s-2)^2} - \frac{3}{s-2}$.

Example 155. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right)$.

Solution. $\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right) = e^{3t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = te^{3t}$.

Example 156. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{(s-3)^2}\right)$.

Solution. It follows from the previous example that $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{(s-3)^2}\right) = u_2(t)(t-2)\,e^{3(t-2)}$.

Example 157. (bonus) Solve the IVP $y'' - 3y' + 2y = e^t$, $y(0) = 0$, $y'(0) = 1$.

Solution. (old style, outline) The characteristic polynomial $D^2 - 3D + 2 = (D-1)(D-2)$. Since there is duplication, we have to look for a particular solution of the form $y_p = Ate^t$. To determine A, we need to plug into the DE (we find $A = -1$). Then, the general solution is $y(t) = Ate^t + C_1e^t + C_2e^{2t}$, and the initial conditions determine C_1 and C_2 (we find $C_1 = -2$ and $C_2 = 2$).

Solution. (Laplace style)

$$\begin{aligned}\mathcal{L}(y''(t)) - 3\mathcal{L}(y'(t)) + 2\mathcal{L}(y(t)) &= \mathcal{L}(e^t)\\ s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) &= \frac{1}{s-1}\\ (s^2 - 3s + 2)Y(s) &= 1 + \frac{1}{s-1} = \frac{s}{s-1}\\ Y(s) &= \frac{s}{(s-1)^2(s-2)}\end{aligned}$$

To find $y(t)$, we again use partial fractions. We find $Y(s) = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-2}$ with coefficients (why?!)

$$C = \frac{s}{(s-1)^2}\bigg|_{s=2} = 2, \quad A = \frac{s}{s-2}\bigg|_{s=1} = -1, \quad B = \frac{\mathrm{d}}{\mathrm{d}s}\frac{s}{s-2}\bigg|_{s=1} = \frac{-2}{(s-2)^2}\bigg|_{s=1} = -2.$$

Finally, $y(t) = \mathcal{L}^{-1}\left(\frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-2}\right) = Ate^t + Be^t + Ce^{2t} = -(t+2)e^t + 2e^{2t}$.

More details on the partial fractions with a repeated root. Above we computed A, B, C so that

$$\frac{s}{(s-1)^2(s-2)} = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-2}.$$

- We can compute C as before by multiplying both sides with $s-2$ and then setting $s = 2$.
- Similarly, we can compute A by multiplying both sides with $(s-1)^2$ and then setting $s = 1$.
- To compute B, multiply both sides by $(s-1)^2$ (as for A) to get

$$\frac{s}{(s-2)} = A + B(s-1) + \frac{C(s-1)^2}{s-2}.$$

Now, we take the derivative on both sides (so that A goes away) to get

$$\frac{-2}{(s-2)^2} = B + \frac{C(2(s-1)(s-2) - (s-1)^2)}{(s-2)^2}$$

and we find B by setting $s = 1$.

Comment. In fact, the term involving C had to drop out when plugging in $s = 1$, even after taking a derivative. That's because, after multiplying with $(s-1)^2$, that term has a double root at $s = 1$. When taking a derivative, it therefore still has a (single) root at $s = 1$.

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Solving systems of DEs using Laplace transforms

We solved the following system in Example 123 using elimination and our method for solving linear DEs with constant coefficients based on characteristic roots.

Example 158. (extra) Solve the system $y_1' = 5y_1 + 4y_2$, $y_2' = 8y_1 + y_2$, $y_1(0) = 0$, $y_2(0) = 1$.

Solution. (using Laplace transforms) $y_1' = 5y_1 + 4y_2$ transforms into $sY_1 - \underbrace{y_1(0)}_{=0} = 5Y_1 + 4Y_2$.

Likewise, $y_2' = 8y_1 + y_2$ transforms into $sY_2 - \underbrace{y_2(0)}_{=1} = 8Y_1 + Y_2$.

The transformed equations are regular equations that we can solve for Y_1 and Y_2.

For instance, by the first equation, $Y_2 = \frac{1}{4}(s-5)Y_1$.

Used in the second equation, we get $\underbrace{-8Y_1 + \frac{1}{4}(s-1)(s-5)Y_1}_{=\frac{1}{4}(s^2-6s-27)=\frac{1}{4}(s+3)(s-9)} = 1$ so that $Y_1 = \frac{4}{(s+3)(s-9)}$.

Hence, the system is solved by $Y_1 = \frac{4}{(s+3)(s-9)}$ and $Y_2 = \frac{1}{4}(s-5)Y_1 = \frac{s-5}{(s+3)(s-9)}$.

As a final step, we need to take the inverse Laplace transform to get $y_1(t) = \mathcal{L}^{-1}(Y_1(s))$ and $y_2(t) = \mathcal{L}^{-1}(Y_2(s))$.

Using partial fractions, $Y_1(s) = \frac{4}{(s+3)(s-9)} = -\frac{1}{3} \cdot \frac{1}{s+3} + \frac{1}{3} \cdot \frac{1}{s-9}$ so that $y_1(t) = -\frac{1}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

Similarly, $Y_2(s) = \frac{s-5}{(s+3)(s-9)} = \frac{2}{3} \cdot \frac{1}{s+3} + \frac{1}{3} \cdot \frac{1}{s-9}$ so that $y_2(t) = \frac{2}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

Solution. (old solution, for comparison) Since $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1$ (from the first eq.), we have $y_2' = \frac{1}{4}y_1'' - \frac{5}{4}y_1'$.

Using these in the second equation, we get $\frac{1}{4}y_1'' - \frac{5}{4}y_1' = 8y_1 + \frac{1}{4}y_1' - \frac{5}{4}y_1$.
Simplified, this is $y_1'' - 6y_1' - 27y_1 = 0$.
This is a homogeneous linear DE with constant coefficients. The characteristic roots are $-3, 9$.
We therefore obtain $y_1 = C_1e^{-3t} + C_2e^{9t}$ as the general solution.
Thus, $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1 = \frac{1}{4}(-3C_1e^{-3t} + 9C_2e^{9t}) - \frac{5}{4}(C_1e^{-3t} + C_2e^{9t}) = -2C_1e^{-3t} + C_2e^{9t}$.
We determine the (unique) values of C_1 and C_2 using the initial conditions:
$y_1(0) = C_1 + C_2 \stackrel{!}{=} 0$
$y_2(0) = -2C_1 + C_2 \stackrel{!}{=} 1$
We solve these two equations and find $C_1 = -\frac{1}{3}$ and $C_2 = \frac{1}{3}$.

The unique solution to the IVP therefore is $y_1(t) = -\frac{1}{3}e^{-3t} + \frac{1}{3}e^{9t}$ and $y_2(t) = \frac{2}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

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