Handling discontinuities with the Laplace transform

Let $u_a(t) = \begin{cases} 1, & \text{if } t \geqslant a, \\ 0, & \text{if } t < a, \end{cases}$ be the **unit step function**. Throughout, we assume that $a \geqslant 0$.

Comment. The special case $u_0(t)$ is also known as the **Heaviside function**, after Oliver Heaviside who, among many other things, coined terms like conductance and impedance. Note that $u_a(t) = u_0(t-a)$.

Example 144. If $a < b$, then $u_a(t) - u_b(t) = \begin{cases} 1, & \text{if } a \leqslant t < b, \\ 0, & \text{otherwise.} \end{cases}$

Comment. See Example 147 for how to write piecewise-defined functions as combinations of unit step functions.

The following is a useful addition to our table of Laplace transforms:

Example 145. **(new entry)** We add the following to our table of Laplace transforms:

$$\begin{aligned} \mathcal{L}(u_a(t) f(t-a)) &= \int_a^\infty e^{-st} f(t-a) \mathrm{d}t = \int_0^\infty e^{-s(\tilde{t}+a)} f(\tilde{t}) \mathrm{d}\tilde{t} \\ &= e^{-as} \int_0^\infty e^{-s\tilde{t}} f(\tilde{t}) \mathrm{d}\tilde{t} = e^{-as} F(s) \end{aligned}$$

Comment. Note that the graph of $u_a(t) f(t-a)$ is the same as $f(t)$ but delayed by a (make a sketch!).

In particular. $\mathcal{L}(u_a(t)) = \frac{e^{-sa}}{s}$

Thus equipped, we can solve differential equations featuring certain kinds of discontinuities.

Note that the DE in our next example describes the motion of a mass on a spring with damping, where the external force is zero except for the time interval $[2,3)$ when we suddenly have a force equal to 5.

Example 146. Determine the Laplace transform of the unique solution to the initial value problem

$$y'' + 5y' + 6y = \begin{cases} 5, & \text{if } 2 \leqslant t < 3, \\ 0, & \text{otherwise,} \end{cases} \qquad y(0) = -4, \quad y'(0) = 8.$$

Solution. First, we observe that the right-hand side of the differential equation can be written as $5(u_2(t) - u_3(t))$. It follows from the Laplace transform table that $\mathcal{L}(u_a(t)) = e^{-as}\frac{1}{s}$ (using the entry for $u_a(t) f(t-a)$ with $f(t) = 1$). Consequently, $\mathcal{L}(5(u_2(t) - u_3(t))) = 5e^{-2s}\frac{1}{s} - 5e^{-3s}\frac{1}{s} = \frac{5}{s}(e^{-2s} - e^{-3s})$.

Taking the Laplace transform of both sides of the DE, we therefore get

$$s^2 Y(s) - s y(0) - y'(0) + 5(sY(s) - y(0)) + 6Y(s) = \frac{5}{s}(e^{-2s} - e^{-3s}),$$

which using the initial values simplifies to

$$(s^2 + 5s + 6) Y(s) + 4s - 8 + 5 \cdot 4 = \frac{5}{s}(e^{-2s} - e^{-3s}).$$

We conclude that the Laplace transform of the unique solution is

$$Y(s) = \frac{1}{s^2 + 5s + 6}\left[\frac{5}{s}(e^{-2s} - e^{-3s}) - 4s - 12\right].$$

First challenge. Take the inverse Laplace transform to find $y(t)$! (See Examples 148 and 149.)

Second challenge. Solve the DE without using Laplace transforms! (First, solve the IVP for $t < 2$ in which case we have no external force. That tells us what $y(2)$ and $y'(2)$ should be. Using these as the new initial conditions, solve the IVP for $t \in [2,3)$. Then, using $y(3)$ and $y'(3)$, solve the IVP for $t \geqslant 3$. In the end, you will have found the solution $y(t)$ in three pieces. On the other hand, the Laplace transform allows us to avoid working piece-by-piece.)

The next example illustrates that any piecewise defined function can be written using a single formula involving step functions. This is based on the simple observation from Example 144 that $u_a(t) - u_b(t)$ is a function which is 1 on the interval $[a, b)$ but zero everywhere else.

Comment. For our present purposes, we don't really care about the precise value of a function at a single point. Specifically, it doesn't really matter which value the function $u_a(t) - u_b(t)$ takes at $t = b$ (technically, the value is 0 but it may as well be 1 since there is a discontinuity at $t = b$).

Example 147. Write $f(t) = \begin{cases} 0, & \text{if } t < 0, \\ t^2, & \text{if } 0 \leqslant t < 1, \\ 3, & \text{if } 1 \leqslant t < 2, \\ \cos(t-2), & \text{if } t \geqslant 2, \end{cases}$ as a combination of unit step functions.

Solution. $f(t) = t^2(u_0(t) - u_1(t)) + 3(u_1(t) - u_2(t)) + \cos(t-2)u_2(t)$

Homework. Compute the Laplace transform of $f(t)$ from here. Note that, for instance, to find $\mathcal{L}(t^2 u_1(t))$, we want to use $\mathcal{L}(u_a(t)f(t-a)) = e^{-sa}F(s)$ with $a = 1$ and $f(t-1) = t^2$. Then, $f(t) = (t+1)^2 = t^2 + 2t + 1$ has Laplace transform $F(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$. Combined, we get $\mathcal{L}(t^2 u_1(t)) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$.

Example 148. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+3}\right)$.

Solution. $\frac{1}{s+3}$ is the Laplace transform of e^{-3t}. Hence, $\frac{e^{-2s}}{s+3}$ is the Laplace transform of e^{-3t} delayed by 2. In other words, $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+3}\right) = u_2(t)e^{-3(t-2)}$.

Comment. Note that this is one of the terms in our solution $Y(s)$ in Example 146 (because $s^2 + 5s + 6 = (s+2)(s+3)$). Can you determine the full inverse Laplace transform of $Y(s)$?

Example 149. Solve the IVP $y'' + 3y' + 2y = f(t)$, $y(0) = y'(0) = 0$ with $f(t) = \begin{cases} 1, & 3 \leqslant t < 4, \\ 0, & \text{otherwise}. \end{cases}$

Solution. First, we write $f(t) = u_3(t) - u_4(t)$. We can now take the Laplace transform of the DE to get

$$s^2 Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) = \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s} = (e^{-3s} - e^{-4s})\frac{1}{s}.$$

Using that $s^2 + 3s + 2 = (s+1)(s+2)$, we find

$$Y(s) = (e^{-3s} - e^{-4s})\frac{1}{s(s+1)(s+2)} = (e^{-3s} - e^{-4s})\left[\frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}\right],$$

where A, B, C are determined by partial fractions (we compute the values below). Taking the inverse Laplace transform of each of the six terms in this product, as in Example 148, we find

$$y(t) = A(u_3(t) - u_4(t)) + B(u_3(t)e^{-(t-3)} - u_4(t)e^{-(t-4)}) + C(u_3(t)e^{-2(t-3)} - u_4(t)e^{-2(t-4)}).$$

If preferred, we can express this as $y(t) = \begin{cases} 0, & \text{if } t < 3, \\ A + Be^{-(t-3)} + Ce^{-2(t-3)}, & \text{if } 3 \leqslant t < 4, \\ B(e^{-(t-3)} - e^{-(t-4)}) + C(e^{-2(t-3)} - e^{-2(t-4)}) & \text{if } t \geqslant 4. \end{cases}$

Finally, $A = \left.\frac{1}{(s+1)(s+2)}\right|_{s=0} = \frac{1}{2}$, $B = \left.\frac{1}{s(s+2)}\right|_{s=-1} = -1$, $C = \left.\frac{1}{s(s+1)}\right|_{s=-2} = \frac{1}{2}$.

Comment. Check that these values make $y(t)$ a continuous function (as it should be for physical reasons).

Example 150. Determine the Laplace transform $\mathcal{L}(e^{rt}u_a(t))$.

Solution. Write $e^{rt}u_a(t) = f(t-a)u_a(t)$ with $f(t) = e^{r(t+a)} = e^{ra}e^{rt}$. Since $F(s) = \mathcal{L}(f(t)) = \frac{e^{ra}}{s-r}$, we have

$$\mathcal{L}(e^{rt}u_a(t)) = e^{-sa}F(s) = \frac{e^{-(s-r)a}}{s-r}.$$

Example 151. (extra practice) Determine the Laplace transform of the unique solution to the initial value problem

$$y'' - 6y' + 5y = \begin{cases} 3e^{-2t}, & \text{if } 1 \leqslant t < 4, \\ 0, & \text{otherwise,} \end{cases} \qquad y(0) = 2, \quad y'(0) = -1.$$

Solution. First, we write the right-hand side of the differential equation as $f(t) := 3e^{-2t}(u_1(t) - u_4(t))$. By Example 150, the Laplace transform of $f(t)$ is $\mathcal{L}(f(t)) = 3\frac{e^{-(s+2)}}{s+2} - 3\frac{e^{-4(s+2)}}{s+2} = \frac{3}{s+2}(e^{-(s+2)} - e^{-4(s+2)})$. Taking the Laplace transform of both sides of the DE, we therefore get

$$s^2Y(s) - sy(0) - y'(0) - 6(sY(s) - y(0)) + 5Y(s) = \frac{3}{s+2}(e^{-(s+2)} - e^{-4(s+2)}),$$

which using the initial values simplifies to

$$(s^2 - 6s + 5)Y(s) - 2s + 13 = \frac{3}{s+2}(e^{-(s+2)} - e^{-4(s+2)}).$$

We conclude that the Laplace transform of the unique solution is

$$Y(s) = \frac{1}{s^2 - 6s + 5}\left[\frac{3}{s+2}(e^{-(s+2)} - e^{-4(s+2)}) + 2s - 13\right].$$