Notes for Lecture 33

The inverse Laplace transform

Theorem 134. (uniqueness of Laplace transforms) If $\mathcal{L}(f_1(t)) = \mathcal{L}(f_2(t))$, then $f_1(t) = f_2(t)$.

Hence, we can recover $f(t)$ from $F(s)$. We write $\mathcal{L}^{-1}(F(s)) = f(t)$.

We say that $f(t)$ is the **inverse Laplace transform** of $F(s)$.

Advanced comment. This uniqueness is true for continuous functions f_1, f_2. It is also true for piecewise continuous functions except at those values of t for which there is a discontinuity. (Note that redefining $f(t)$ at a single point, will not change its Laplace transform.)

Example 135. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{5}{s+3}\right)$.

Solution. In other words, if $F(s) = \frac{5}{s+3}$, what is $f(t)$?

$\mathcal{L}^{-1}\left(\frac{5}{s+3}\right) = 5\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = 5e^{-3t}$

Example 136. (extra) Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{3s-7}{s^2+4}\right)$.

Solution. In other words, if $F(s) = \frac{3s-7}{s^2+4}$, what is $f(t)$?

$F(s) = 3\frac{s}{s^2+2^2} - \frac{7}{2}\frac{2}{s^2+2^2}$. Hence, $f(t) = 3\cos(2t) - \frac{7}{2}\sin(2t)$.

Example 137. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right)$.

Solution. Note that $s^2 - s - 6 = (s-3)(s+2)$. We use **partial fractions** to write $-\frac{6s-23}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$. We find the coefficients (see brief review below) as

$$A = -\frac{6s-23}{s+2}\bigg|_{s=3} = 1, \quad B = -\frac{6s-23}{s-3}\bigg|_{s=-2} = -7.$$

Hence $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-3} - \frac{7}{s+2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) - 7\mathcal{L}^{-1}\left(\frac{7}{s+2}\right) = e^{3t} - 7e^{-2t}$.

Review. In order to find A, we multiply $-\frac{6s-23}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$ by $s-3$ to get $-\frac{6s-23}{s+2} = A + \frac{B(s-3)}{s+2}$. We then set $s=3$ to find A as above.

Comment. Compare with Example 131 where we considered the same functions.

Example 138. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{s+13}{s^2-s-2}\right)$.

Solution. Note that $s^2 - s - 2 = (s-2)(s+1)$. We use partial fractions to write $\frac{s+13}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}$. We find the coefficients as

$$A = \frac{s+13}{s+1}\bigg|_{s=2} = 5, \quad B = \frac{s+13}{s-2}\bigg|_{s=-1} = -4.$$

Hence $\mathcal{L}^{-1}\left(\frac{s+13}{s^2-s-2}\right) = \mathcal{L}^{-1}\left(\frac{5}{s+1} - \frac{4}{s-2}\right) = 5e^{t} - 4e^{2t}$.

Solving simple DEs using the Laplace transform

In the following examples, we write $Y(s)$ for the Laplace transform of $y(t)$.

Recall from our Laplace transform table that this implies that the Laplace transform of $y'(t)$ is $sY(s) - y(0)$.

Example 139. Solve the (very simple) IVP $y'(t) - 2y(t) = 0$, $y(0) = 7$.

At this point, you might be able to "see" right away that the unique solution is $y(t) = 7e^{2t}$.

Solution. (old style) The characteristic root is 2, so that the general solution is $y(t) = Ce^{2t}$. Using the initial condition, we find that $C = 7$, so that $y(t) = 7e^{2t}$.

Solution. (Laplace style) $y' - 2y = 0$ transforms into

$$\mathcal{L}(y'(t) - 2y(t)) = \mathcal{L}(y'(t)) - 2\mathcal{L}(y(t)) = sY(s) - y(0) - 2Y(s) = (s-2)Y(s) - 7 = 0.$$

This is an algebraic equation for $Y(s)$. It follows that $Y(s) = \frac{7}{s-2}$. By inverting the Laplace transform, we conclude that $y(t) = 7e^{2t}$.

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