The Laplace transform

Definition 127. The **Laplace transform** of a function $f(t)$, $t \geqslant 0$, is defined as the new function

$$F(s) = \int_0^\infty e^{-st} f(t) \mathrm{d}t.$$

We also write $\mathcal{L}(f(t)) = F(s)$.

Note that, in order for the integral to exist, $f(t)$ should be, say, piecewise continuous and of at most exponential growth. That's true for most of the functions we are interested in (and so we will not dwell on this issue).

$f(t)$	$F(s)$
e^{at}	$\frac{1}{s-a}$
1	$\frac{1}{s}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - s f(0) - f'(0)$

First entries in the Laplace transform table

In this section, we will discuss and obtain the entries in the table of the most basic Laplace transforms that we compiled after Definition 127.

Example 128. Show that $\mathcal{L}(e^{at}) = \frac{1}{s-a}$.

In particular, in the special case $a = 0$, we have $\mathcal{L}(1) = \frac{1}{s}$.

Solution. $\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at} \mathrm{d}t = \int_0^\infty e^{(a-s)t} \mathrm{d}t = \left[\frac{1}{a-s} e^{(a-s)t}\right]_{t=0}^\infty = 0 - \frac{1}{a-s} = \frac{1}{s-a}$

Comment. Note that we needed $a - s < 0$ in order for the integral to converge. Hence the Laplace transform has domain $s > a$. (During this introduction, we will not care too much about these technical details.)

In particular. Note that setting $a = 0$ shows that $\mathcal{L}(1) = \frac{1}{s}$.

Example 129. (linearity) Show that $\mathcal{L}(c_1 f_1(t) + c_2 f_2(t)) = c_1 F_1(s) + c_2 F_2(s)$.

This means that the Laplace transform is a **linear operator** (like the derivative or the integral).

Solution.

$$\begin{aligned}
\mathcal{L}(c_1 f_1(t) + c_2 f_2(t)) &= \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t)) \mathrm{d}t \\
&= c_1 \underbrace{\int_0^\infty e^{-st} f_1(t) \mathrm{d}t}_{F_1(s)} + c_2 \underbrace{\int_0^\infty e^{-st} f_2(t) \mathrm{d}t}_{F_2(x)}
\end{aligned}$$

Example 130. (extra) Show that $\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$ and $\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2}$.

Solution. By Euler's identity, $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$. Hence, by linearity,

$$\mathcal{L}(e^{i\omega t}) = \mathcal{L}(\cos(\omega t)) + i\mathcal{L}(\sin(\omega t)).$$

On the other hand,

$$\mathcal{L}(e^{i\omega t}) = \frac{1}{s - i\omega} = \frac{s + i\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}.$$

Matching real and imaginary parts, we find $\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$ and $\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2}$.

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Example 131. Determine $\mathcal{L}(e^{3t} - 7e^{-2t})$.

Solution. $\mathcal{L}(e^{3t} - 7e^{-2t}) = \mathcal{L}(e^{3t}) - 7\mathcal{L}(e^{-2t}) = \frac{1}{s-3} - \frac{7}{s+2}$

Comment. Note that, once we write $\frac{1}{s-3} - \frac{7}{s+2} = -\frac{6s-23}{s^2-s-6}$ it is no longer visibly clear which function we have taken the Laplace transform of. We discuss reversing this process in the next section.

Example 132. (extra) Determine $\mathcal{L}(3\cos(2t) - 5\sin(2t))$.

Solution. $\mathcal{L}(3\cos(2t) - 5\sin(2t)) = 3\mathcal{L}(\cos(2t)) - 5\mathcal{L}(\sin(2t)) = 3\frac{s}{s^2+4} - 5\frac{2}{s^2+4} = \frac{3s-10}{s^2+4}$

Example 133. Show that $\mathcal{L}(f'(t)) = sF(s) - f(0)$.

Solution. Using integration by parts,

$$\mathcal{L}(f'(t)) = \int_0^\infty e^{-st} f'(t)\mathrm{d}t = \left[e^{-st} f(t)\right]_{t=0}^\infty + \int_0^\infty s e^{-st} f(t)\mathrm{d}t = sF(s) - f(0).$$

Higher derivatives. In order to obtain the Laplace transform of higher derivatives, we can iterate. For instance,

$$\mathcal{L}(f''(t)) = s\mathcal{L}(f'(t)) - f'(0) = s[sF(s) - f(0)] - f'(0) = s^2F(s) - sf(0) - f'(0).$$

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