Example 123. Determine the general solution to $y_1' = 5y_1 + 4y_2 + e^{2x}$, $y_2' = 8y_1 + y_2$.

Solution. From the second equation it follows that $y_1 = \frac{1}{8}(y_2' - y_2)$. Using this in the first equation, we get $\frac{1}{8}(y_2'' - y_2') = \frac{5}{8}(y_2' - y_2) + 4y_2 + e^{2x}$. After multiplying with 8, this is $y_2'' - y_2' = 5(y_2' - y_2) + 32y_2 + 8e^{2x}$.
Simplified, this is $y_2'' - 6y_2' - 27y_2 = 8e^{2x}$, which is an inhomogeneous linear DE with constant coefficients which know how to solve:

- Since the "old" roots are $-3, 9$, while the "new" root is 2, there must a particular solution of the form $y_p = Ce^{2x}$. Plugging this y_p into the DE, we get $y_p'' - 6y_p' - 27y_p = (4 - 6 \cdot 2 - 27)Ce^{2x} = -35Ce^{2x} \overset{!}{=} 8e^{2x}$. Hence, $C = -\frac{8}{35}$.
- We therefore obtain $y_2 = -\frac{8}{35}e^{2x} + C_1e^{-3x} + C_2e^{9x}$ as the general solution to the inhomogeneous DE.

We can then determine y_1 as

$$\begin{aligned} y_1 &= \frac{1}{8}(y_2' - y_2) \\ &= \frac{1}{8}\left(-\frac{16}{35}e^{2x} - 3C_1e^{-3x} + 9C_2e^{9x} + \frac{8}{35}e^{2x} - C_1e^{-3x} - C_2e^{9x}\right) \\ &= -\frac{1}{35}e^{2x} - \frac{1}{2}C_1e^{-3x} + C_2e^{9x}. \end{aligned}$$

Solution. (alternative) We can also start with $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1 - \frac{1}{4}e^{2x}$ (from the first equation), although the algebra will require a little more work. In that case, we have $y_2' = \frac{1}{4}y_1'' - \frac{5}{4}y_1' - \frac{1}{2}e^{2x}$. Using this in the second equation, we get $\frac{1}{4}y_1'' - \frac{5}{4}y_1' - \frac{1}{2}e^{2x} = 8y_1 + \frac{1}{4}y_1' - \frac{5}{4}y_1 - \frac{1}{4}e^{2x}$.
Simplified, this is $y_1'' - 6y_1' - 27y_1 = e^{2x}$, which is an inhomogeneous linear DE with constant coefficients which know how to solve:

- Since the "old" roots are $-3, 9$, while the "new" root is 2, there must a particular solution of the form $y_p = Ce^{2x}$. Plugging this y_p into the DE, we get $y_p'' - 6y_p' - 27y_p = (4 - 6 \cdot 2 - 27)Ce^{2x} = -35Ce^{2x} \overset{!}{=} e^{2x}$. Hence, $C = -\frac{1}{35}$.
- We therefore obtain $y_1 = -\frac{1}{35}e^{2x} + C_1e^{-3x} + C_2e^{9x}$ as the general solution to the inhomogeneous DE.

We can then determine y_2 as

$$\begin{aligned} y_2 &= \frac{1}{4}y_1' - \frac{5}{4}y_1 - \frac{1}{4}e^{2x} \\ &= \frac{1}{4}\left(-\frac{2}{35}e^{2x} - 3C_1e^{-3x} + 9C_2e^{9x}\right) - \frac{5}{4}\left(-\frac{1}{35}e^{2x} + C_1e^{-3x} + C_2e^{9x}\right) - \frac{1}{4}e^{2x} \\ &= -\frac{8}{35}e^{2x} - 2C_1e^{-3x} + C_2e^{9x}. \end{aligned}$$

Important. Make sure you can explain why both of our solutions are equivalent!

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Example 124.

(a) Determine the general solution to $y_1' = 5y_1 + 4y_2$, $y_2' = 8y_1 + y_2$.

Comment. In matrix form, with $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, this is $\boldsymbol{y}' = \begin{bmatrix} 5 & 4 \\ 8 & 1 \end{bmatrix} \boldsymbol{y}$.

(b) Solve the IVP $y_1' = 5y_1 + 4y_2$, $y_2' = 8y_1 + y_2$, $y_1(0) = 0$, $y_2(0) = 1$.

Solution.

(a) Note that this is the homogeneous system corresponding to the previous problem. It therefore follows from our previous solution (the latter one) that $y_1 = C_1 e^{-3x} + C_2 e^{9x}$ and $y_2 = -2C_1 e^{-3x} + C_2 e^{9x}$ is the general solution of the homogeneous system.

(b) We already have the general solutions y_1, y_2 to the two DEs. We need to determine the (unique) values of C_1 and C_2 to match the initial conditions: $y_1(0) = C_1 + C_2 \overset{!}{=} 0$, $y_2(0) = -2C_1 + C_2 \overset{!}{=} 1$

We solve these two equations and find $C_1 = -\frac{1}{3}$ and $C_2 = \frac{1}{3}$.

The unique solution to the IVP therefore is $y_1 = -\frac{1}{3}e^{-3x} + \frac{1}{3}e^{9x}$ and $y_2 = \frac{2}{3}e^{-3x} + \frac{1}{3}e^{9x}$.