Review. Modeling two connected fluid tanks

Higher-order linear DEs as first-order systems

Example 120. Write the (second-order) differential equation $y'' = 2y' + 5y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$ and $y_2 = y'$. Then $y'' = 2y' + 5y$ becomes $y_2' = 2y_2 + 5y_1$.

Therefore, $y'' = 2y' + 5y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = 5y_1 + 2y_2 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 5 & 2 \end{bmatrix} \boldsymbol{y}$.

Comment. This illustrates why we might care about systems of DEs, even if we work with only one function.

Example 121. Write the (third-order) differential equation $y''' = 3y'' - 2y' + 4y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' = 3y'' - 2y' + 4y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = 4y_1 - 2y_2 + 3y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -2 & 3 \end{bmatrix} \boldsymbol{y}$.

Example 122. Consider the following system of (second-order) initial value problems:

$$\begin{array}{l} y_1'' = 2y_1' - 3y_2' + 7y_2 \\ y_2'' = 4y_1' + y_2' - 5y_1 \end{array} \qquad y_1(0) = 2,\ y_1'(0) = 3,\ y_2(0) = -1,\ y_2'(0) = 1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

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