Systems of differential equations

Modeling two connected fluid tanks

Example 117. Consider two brine tanks. Tank T_1 contains 24gal water containing 3lb salt, and tank T_2 contains 9gal pure water.

- T_1 is being filled with 54gal/min water containing 0.5lb/gal salt.
- 72gal/min well-mixed solution flows out of T_1 into T_2.
- 18gal/min well-mixed solution flows out of T_2 into T_1.
- Finally, 54gal/min well-mixed solution is leaving T_2.

How much salt is in the tanks after t minutes?

Solution. Note that the amount of water in each tank is constant because the flows balance each other.
Let $y_i(t)$ denote the amount of salt (in lb) in tank T_i after time t (in min). In the time interval $[t, t+\Delta t]$:
$\Delta y_1 \approx 54 \cdot \frac{1}{2} \cdot \Delta t - 72 \cdot \frac{y_1}{24} \cdot \Delta t + 18 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_1' = 27 - 3y_1 + 2y_2$. Also, $y_1(0) = 3$.
$\Delta y_2 \approx 72 \cdot \frac{y_1}{24} \cdot \Delta t - 72 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_2' = 3y_1 - 8y_2$. Also, $y_2(0) = 0$.

One strategy to solve this system is to first combine the two DEs to get a single equation for y_1.

- From the first DE, we get $y_2 = \frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2}$.
- Using this in the second DE, we obtain $\left(\frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2}\right)' = 3y_1 - 8\left(\frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2}\right)$.
  Simplified, this is $y_1'' + 11y_1' + 18y_1 = 216$.
- We already have the initial condition $y_1(0) = 3$. We get a second one by combining $y_2 = \frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2}$ with $y_2(0) = 0$ to get $0 = y_2(0) = \frac{1}{2}y_1'(0) + \frac{3}{2}y_1(0) - \frac{27}{2} = \frac{1}{2}y_1'(0) - 9$, which simplifies to $y_1'(0) = 18$.
- The IVP $y_1'' + 11y_1' + 18y_1 = 216$ with initial conditions $y_1(0) = 3$ and $y_1'(0) = 18$ is one that we can solve!
  - The general solution of the corresponding homogeneous equation is $y_h = C_1 e^{-2t} + C_2 e^{-9t}$.
  - The simplest particular solution is of the form $y_p = C$. Plugging into the DE, we find $y_p = \frac{216}{18} = 12$.
  - Hence, the general solution to the (inhomogeneous) DE is $y(x) = 12 + C_1 e^{-2t} + C_2 e^{-9t}$.
    We then use the initial conditions $y(0) = 12 + C_1 + C_2 \overset{!}{=} 3$, $y'(0) = -2C_1 - 9C_2 \overset{!}{=} 18$ to find that for the unique solution of the IVP $C_1 = -9$, $C_2 = 0$.

  It has the unique solution $y_1(t) = 12 - 9e^{-2t}$.
- It follows that $y_2 = \frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2} = \frac{9}{2} - \frac{9}{2}e^{-2t}$.

Note. We could have found a particular solution with less calculations by observing (looking at "old" and "new" roots) that there must be a solution of the form $\boldsymbol{y}_p(t) = \boldsymbol{a}$. We can then find $\boldsymbol{a}$ by plugging into the differential equation. However, noticing that, for a constant solution, each tank has to have a constant concentration of 0.5lb/gal of salt, we find $\boldsymbol{y}_p(t) = \begin{bmatrix} 12 \\ 4.5 \end{bmatrix}$.

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Example 118. Write the system of equations

$$\begin{array}{rcll} y_1' &=& -3y_1 + 2y_2 + 27 & \qquad y_1(0) = 3 \\ y_2' &=& 3y_1 - 8y_2 & \qquad y_2(0) = 0 \end{array}$$

from the previous problem in matrix-vector notation.

Solution. If we write $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then the system becomes

$$\boldsymbol{y}' = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 27 \\ 0 \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

Advanced comment. Here, we only use the matrix-vector notation as a device for writing the system of equations in a more compact form. However, it turns out that the matrix-vector notation makes certain techniques more transparent (just like writing a system of equations in the form $A\boldsymbol{x} = \boldsymbol{b}$ suggests introducing the matrix inverse to simply write $\boldsymbol{x} = A^{-1}\boldsymbol{b}$). For instance, the unique solution to a homogeneous linear system $\boldsymbol{y}' = M\boldsymbol{y}$ (where M is a matrix with constant entries) with initial condition $\boldsymbol{y}(0) = \boldsymbol{c}$ can be expressed as $\boldsymbol{y}(x) = e^{Mx}\boldsymbol{c}$, just as in the case of a single linear DE. Here, e^{Mx} is the **matrix exponential**. This will be one of the topics discussed in both Differential Equations II and Linear Algebra II.

Example 119. (extra) Three brine tanks T_1, T_2, T_3.

T_1 contains 20gal water with 10lb salt, T_2 40gal pure water, T_3 50gal water with 30lb salt.

T_1 is filled with 10gal/min water with 2lb/gal salt. 10gal/min well-mixed solution flows out of T_1 into T_2. Also, 10gal/min well-mixed solution flows out of T_2 into T_3. Finally, 10gal/min well-mixed solution is leaving T_3. How much salt is in the tanks after t minutes?

Solution. Let $y_i(t)$ denote the amount of salt (in lb) in tank T_i after time t (in min).

In the time interval $[t, t + \Delta t]$:

$\Delta y_1 \approx 10 \cdot 2 \cdot \Delta t - 10 \frac{y_1}{20} \cdot \Delta t$, so $y_1' = 20 - \frac{1}{2} y_1$. Also, $y_1(0) = 10$.

$\Delta y_2 \approx 10 \cdot \frac{y_1}{20} \cdot \Delta t - 10 \frac{y_2}{40} \cdot \Delta t$, so $y_2' = \frac{1}{2} y_1 - \frac{1}{4} y_2$. Also, $y_2(0) = 0$.

$\Delta y_3 \approx 10 \cdot \frac{y_2}{40} \cdot \Delta t - 10 \frac{y_3}{50} \cdot \Delta t$, so $y_3' = \frac{1}{4} y_2 - \frac{1}{5} y_3$. Also, $y_3(0) = 30$.

Using matrix notation and writing $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, this is $\boldsymbol{y}' = \begin{bmatrix} -1/2 & 0 & 0 \\ 1/2 & -1/4 & 0 \\ 0 & 1/4 & -1/5 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 10 \\ 0 \\ 30 \end{bmatrix}$.

We can actually solve this IVP!

[Do it! Start by finding y_1 from the first DE, then move on to $y_2 \ldots$]

Here, we content ourselves with finding a particular solution (and ignoring the initial conditions). The method of undetermined coefficients tells us that there is a solution of the form $\boldsymbol{y}_p(t) = \boldsymbol{a}$. Of course, we can find $\boldsymbol{a}$ by plugging into the differential equation. However, noticing that, for a constant solution, each tank has to have a concentration of 2lb/gal of salt, we find $\boldsymbol{y}_p = (40, 80, 100)$ without calculation.