External forces plus damping

In the presence of both damping and a periodic external force, the motion $y(t) = y_{\text{tr}} + y_{\text{sp}}$ of a mass on a spring splits into **transient motion** y_{tr} (with $y_{\text{tr}}(t) \to 0$ as $t \to \infty$) and **steady periodic oscillations** y_{sp}. The following example spells this out.

Comment. Note that y_{sp} will correspond to the simplest particular solution, while y_{tr} corresponds to the solution of the corresponding homogeneous system (where we have no external force).

Example 113. A forced mechanical oscillator is described by $2y'' + 2y' + y = 10\sin(t)$. As $t \to \infty$, what are the period and the amplitude of the resulting steady periodic oscillations?

Solution. The "old" roots are $\frac{1}{4}(-2 \pm \sqrt{4-8}) = -\frac{1}{2} \pm \frac{1}{2}i$. Accordingly, the system without external force is underdamped (because of the $\pm i/2$ the solutions will involve oscillations).

The "new" roots are $\pm i$ so that there must be a particular solution $y_p = A\cos(t) + B\sin(t)$ with coefficients A, B that we need to determine by plugging into the DE. This results in $A = -4$ and $B = -2$ (do it!).

Hence, the general solution is $y(t) = \underbrace{-4\cos(t) - 2\sin(t)}_{y_{\text{sp}}} + \underbrace{e^{-t/2}\left(C_1\cos\left(\frac{t}{2}\right) + C_2\sin\left(\frac{t}{2}\right)\right)}_{y_{\text{tr}} \to 0 \text{ as } t \to \infty}$.

The period of $y_{\text{sp}} = -4\cos(t) - 2\sin(t)$ is 2π and the amplitude is $\sqrt{(-4)^2 + (-2)^2} = \sqrt{20}$.

Comment. Using the polar coordinates $(-4, -2) = \sqrt{20}(\cos\alpha, \sin\alpha)$ where $\alpha = \tan^{-1}(1/2) + \pi \approx 3.605$, we can alternatively express the steady periodic oscillations as $y_{\text{sp}} = -4\cos(t) - 2\sin(t) = \sqrt{20}(\cos(t - \alpha))$.

Example 114. A forced mechanical oscillator is described by $y'' + 5y' + 6y = 2\cos(3t)$. What are the (circular) frequency and the amplitude of the resulting steady periodic oscillations?

Solution. The "old" roots are $-2, -3$. Accordingly, the system without external force is overdamped (the solutions will not involve oscillations).

The "new" roots are $\pm 3i$ so that there must be a particular solution $y_p = A\cos(3t) + B\sin(3t)$ with coefficients A, B that we need to determine by plugging into the DE. To do so, we compute $y_p' = -3A\sin(3t) + 3B\cos(3t)$ as well as $y_p'' = -9A\cos(3t) - 9B\sin(3t)$.

$$\begin{aligned}
y_p'' + 5y_p' + 6y_p &= (-9A\cos(3t) - 9B\sin(3t)) + 5(-3A\sin(3t) + 3B\cos(3t)) + 6(A\cos(3t) + B\sin(3t)) \\
&= (-9A + 15B + 6A)\cos(3t) + (-9B - 15A + 6B)\sin(3t) \\
&\overset{!}{=} 2\cos(3t)
\end{aligned}$$

This results in the two equations $-3A + 15B = 2$ and $-3B - 15A = 0$, which we solve to find $A = -\frac{1}{39}$ and $B = \frac{5}{39}$.

The general solution is $y(t) = \underbrace{-\frac{1}{39}\cos(3t) + \frac{5}{39}\sin(3t)}_{y_{\text{sp}}} + \underbrace{C_1 e^{-2t} + C_2 e^{-3t}}_{y_{\text{tr}} \to 0 \text{ as } t \to \infty}$.

The frequency of $y_{\text{sp}} = -\frac{1}{39}\cos(3t) + \frac{5}{39}\sin(3t)$ is 3 and the amplitude is $\sqrt{\left(-\frac{1}{39}\right)^2 + \left(\frac{5}{39}\right)^2} = \sqrt{\frac{2}{117}}$.

Armin Straub
straub@southalabama.edu

Example 115. Find the steady periodic solution to $y'' + 2y' + 5y = \cos(\lambda t)$. What is the amplitude of the steady periodic oscillations? For which λ is the amplitude maximal?

Solution. The "old" roots are $-1 \pm 2i$.

[Not really needed, because positive damping prevents duplication; can you see it?]

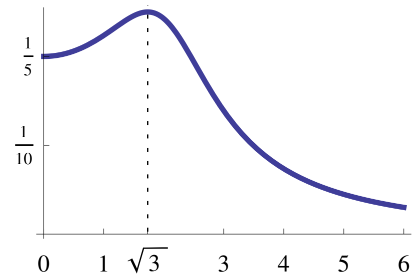
Hence, $y_{\mathrm{sp}} = A\cos(\lambda t) + B\sin(\lambda t)$ and to find A, B we need to plug into the DE.

Doing so, we find $A = \frac{5-\lambda^2}{(5-\lambda^2)^2 + 4\lambda^2}$, $B = \frac{2\lambda}{(5-\lambda^2)^2 + 4\lambda^2}$.

Thus, the amplitude of y_{sp} is $r(\lambda) = \sqrt{A^2 + B^2} = \frac{1}{\sqrt{(5-\lambda^2)^2 + 4\lambda^2}}$.

The function $r(\lambda)$ is sketched to the right. It has a maximum at $\lambda = \sqrt{3}$ at which the amplitude is unusually large (well, here it is not very pronounced). We say that **practical resonance** occurs for $\lambda = \sqrt{3}$.

[For comparison, without damping, (pure) resonance occurs for $\lambda = \sqrt{5}$.]



Example 116. (homework) A car is going at constant speed v on a washboard road surface ("bumpy road") with height profile $y(s) = a\cos\left(\frac{2\pi s}{L}\right)$. Assume that the car oscillates vertically as if on a spring (no dashpot). Describe the resulting oscillations.

Solution. With x as in the sketch, the spring is stretched by $x - y$. Hence, by Hooke's and Newton's laws, its motion is described by $mx'' = -k(x-y)$.

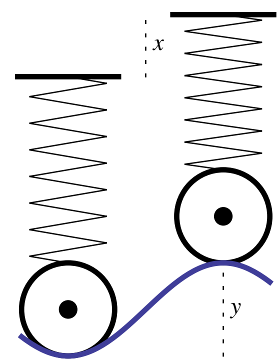
At constant speed, $s = vt$ and we obtain the DE $mx'' + kx = ky = ka\cos\left(\frac{2\pi v t}{L}\right)$, which is inhomogeneous linear with constant coefficients. Let's solve it.

"Old" roots: $\pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$. $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency.

"New" roots: $i\frac{2\pi v}{L} = \pm i\omega$. $\omega = \frac{2\pi v}{L}$ is the external frequency.

Case 1: $\boldsymbol{\omega \neq \omega_0}$. Then a particular solution is $x_p = b_1\cos(\omega t) + b_2\sin(\omega t) = A\cos(\omega t - \alpha)$ for unique values of b_1, b_2 (which we do not compute here). The general solution is of the form $x = x_p + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$.

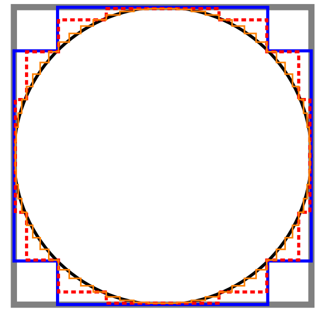
Case 2: $\boldsymbol{\omega = \omega_0}$. Then a particular solution is $x_p = t[b_1\cos(\omega t) + b_2\sin(\omega t)] = At\cos(\omega t - \alpha)$ for unique values of b_1, b_2 (which we do not compute). Note that the amplitude in x_p increases without bound; the same is true for the general solution $x = x_p + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$. This phenomenon is called resonance; it occurs if an external frequency matches a natural frequency.



The first "car" is assumed to be in equilibrium.

Armin Straub
straub@southalabama.edu

(A Halloween scare!) π is the perimeter of a circle enclosed in a square with edge length 1. The perimeter of the square is 4, which approximates π. To get a better approximation, we "fold" the vertices of the square towards the circle (and get the blue polygon). This construction can be repeated for even better approximations and, in the limit, our shape will converge to the true circle. At each step, the perimeter is 4, so we conclude that $\pi = 4$, contrary to popular belief.



Can you pin-point the fallacy in this argument?

(We are not doing something completely silly! For instance, the areas of our approximations do converge to $\pi/4$, the area of the circle.)

The "solution" is below...

($\pi = 4$, "solution")

We are constructing curves c_n with the property that $c_n \to c$ where c is the circle. This convergence can be understood, for instance, in the same sense $\|c_n - c\| \to 0$ with the norm measuring the maximum distance between the two curves.

Since $c_n \to c$ we then wanted to conclude that $\text{perimeter}(c_n) \to \text{perimeter}(c)$, leading to $4 \to \pi$.

However, in order to conclude from $x_n \to x$ that $f(x_n) \to f(x)$ we need that f is continuous (at x)!!

The "function" perimeter, however, is not continuous. In words, this means that (as we see in this example) curves can be arbitrarily close, yet have very different arc length.

We can dig a little deeper: as we learned in Calculus II, the arc length of a function $y = f_n(x)$ for $x \in [a, b]$ is

$$\int_a^b \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \int_a^b \sqrt{1 + f_n'(x)^2}\,\mathrm{d}x.$$

Observe that this involves $f_n'(x)$. Try to see why the operator D that sends f to f' is not continuous with respect to the distance induced by the norm

$$\|f\| = \left(\int_a^b f(x)^2 \mathrm{d}x\right)^{1/2}.$$

In words, two functions f and g can be arbitrarily close, yet have very different derivatives f' and g'.

That's a huge issue in **functional analysis**, which is the generalization of linear algebra to infinite dimensional spaces (like the space of all differentiable functions). The linear operators ("matrices") on these spaces frequently fail to be continuous.

Armin Straub
straub@southalabama.edu