Notes for Lecture 24

Wed, 10/25/2023

The amplitude of oscillations

Example 101. If (r, α) are the **polar coordinates** for (A, B), then

$$A\cos(\omega t) + B\sin(\omega t) = r\cos(\omega t - \alpha).$$

In particular, $A\cos(\omega t) + B\sin(\omega t)$ is periodic with **amplitude** $r = \sqrt{A^2 + B^2}$.

ω is the (circular) **frequency** and α is called the **phase angle**.

Why? Both sides solve

$$y'' + \omega y = 0.$$

The LHS has initial values $y(0) = A$ and $y'(0) = \omega B$, the RHS has $y(0) = r\cos(\alpha)$ and $y'(0) = r\omega\sin(\alpha)$. Hence, the two are equal if $A = r\cos(\alpha)$ and $B = r\sin(\alpha)$.

Alternatively. If you like trig identities, this follows from:
$A\cos(\omega t) + B\sin(\omega t) = r(\cos(\alpha)\cos(\omega t) + \sin(\alpha)\sin(\omega t)) = r\cos(\omega t - \alpha)$.

Review. How to calculate the polar coordinates (r, α) for (A, B)?

We need to find $r \geqslant 0$ and $\alpha \in [0, 2\pi)$ such that $(A, B) = r(\cos\alpha, \sin\alpha)$.
Hence, $r = \sqrt{A^2 + B^2}$ and α is determined by $\cos(\alpha) = \frac{A}{r}$ and $\sin(\alpha) = \frac{B}{r}$.
In particular, $\tan(\alpha) = \frac{B}{A}$ and, if careful, we can compute α using $\tan^{-1}$ as

$$\alpha = \tan^{-1}\left(\frac{B}{A}\right) + \begin{cases} 0, & \text{if } (A, B) \text{ in first quadrant}, \\ 2\pi, & \text{if } (A, B) \text{ in fourth quadrant}, \\ \pi, & \text{otherwise}. \end{cases}$$

Example 102. (again) What is the period and the amplitude of the oscillations $\cos(4t) - 3\sin(4t)$?

Solution. The period is $\frac{2\pi}{4} = \frac{\pi}{2}$.

The amplitude is $\sqrt{1^2 + (-3)^2} = \sqrt{10}$.

Example 103. The motion of a mass on a spring is described by $5y'' + 2y = 0$, $y(0) = 3$, $y'(0) = -1$. What is the period and the amplitude of the resulting oscillations?

Solution. The characteristic roots are $\pm i\omega$ with $\omega = \sqrt{\frac{2}{5}}$. The general solution is $y(t) = A\cos(\omega t) + B\sin(\omega t)$.

The period of the oscillations therefore is $\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{5}{2}} = \pi\sqrt{10}$.

To meet the initial conditions, we need $y(0) = A \stackrel{!}{=} 3$ and $y'(0) = \omega B \stackrel{!}{=} -1$. The latter implies $B = -\frac{1}{\omega} = -\sqrt{\frac{5}{2}}$.
Hence, the amplitude of the oscillations is $\sqrt{A^2 + B^2} = \sqrt{3^2 + \frac{5}{2}} = \sqrt{\frac{23}{2}}$.

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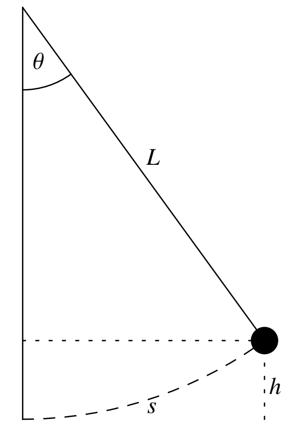
Application: motion of a pendulum

Example 104. Show that the motion of an ideal pendulum is described by

$$L\theta'' + g\sin(\theta) = 0,$$

where θ is the angular displacement and L is the length of the pendulum.

And, as usual, g is acceleration due to gravity.



For short times and small angles, this motion is approximately described by

$$L\theta'' + g\theta = 0.$$

This is because, if θ is small, then $\sin(\theta) \approx \theta$. For instance, for $\theta = 15°$ the error $\theta - \sin\theta$ is about 1%.

Solution. (Newton's second law) The tangential component of the gravitational force is $F = -\sin\theta \cdot mg$. Combining this with Newton's second law, according to which $F = ma = mL\theta''$ (note that $a = s''$ where $s = L\theta$), we obtain the claimed DE.

Solution. (conservation of energy) Alternatively, we can use conservation of energy to derive the DE. Again, we assume the string to be massless, and let m be the swinging mass. Let s and h be as in the sketch above.

The velocity (more accurately, the speed) of the mass is $v = \frac{\mathrm{d}s}{\mathrm{d}t} = L\frac{\mathrm{d}\theta}{\mathrm{d}t}$.

Its kinetic energy therefore is $T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2$.

On the other hand, the potential energy is $V = mgh = mgL(1-\cos\theta)$ (weight mg times height h).

By the principle of conservation of energy, the sum of these is constant: $T + V = \text{const}$

Taking the time derivative, this becomes $\frac{1}{2}mL^2 2\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + mgL\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$. Cancelling terms, we obtain the DE.

Example 105. The motion of a pendulum is described by $\theta'' + 9\theta = 0$, $\theta(0) = 1/4$, $\theta'(0) = 0$. What is the period and the amplitude of the resulting oscillations?

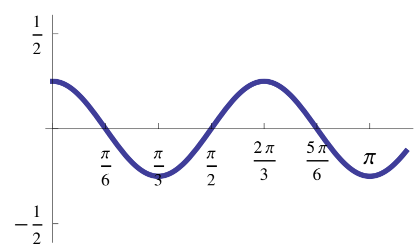
Solution. The roots of the characteristic polynomial are $\pm 3i$.

Hence, $\theta(t) = A\cos(3t) + B\sin(3t)$. $\theta(0) = A = 1/4$. $\theta'(0) = 3B = 0$.

Therefore, the solution is $\theta(t) = 1/4\cos(3t)$.

Hence, the period is $2\pi/3$ and the amplitude is $1/4$.

Comment. $1/4$ is about $14.3°$.



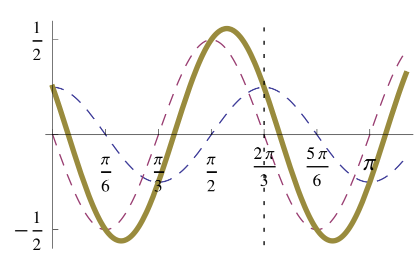
Example 106. The motion of a pendulum is described by $\theta'' + 9\theta = 0$, $\theta(0) = 1/4$, $\theta'(0) = -\frac{3}{2}$ ("initial kick"). What is the period and the amplitude of the resulting oscillations?

Solution. This time, $\theta(0) = A = 1/4$. $\theta'(0) = 3B = -3/2$.

Therefore, the solution is $\theta(t) = \frac{1}{4}\cos(3t) - \frac{1}{2}\sin(3t)$.

Hence, the period is $2\pi/3$ and the amplitude is $\sqrt{\frac{1}{4^2} + \frac{1}{2^2}} = \frac{\sqrt{5}}{4} \approx 0.559$.

Comment. Using polar coordinates, we get $\theta(t) = \frac{\sqrt{5}}{4}\cos(3t - \alpha)$ with phase angle $\alpha = \tan^{-1}(-2) + 2\pi \approx 5.176$.



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