Review. The method of undetermined coefficients versus variation of parameters.

A closer look at second-order linear DEs

Application: motion of a mass on a spring

Example 99. The motion of a mass m attached to a spring is described by

$$my'' + ky = 0$$

where y is the displacement from the equilibrium position and $k > 0$ is the spring constant.

Why? This follows from Hooke's law $F = -ky$ combined with Newton's second law $F = ma = my''$. (Note that the minus sign is needed because the force on the mass is in direction opposite to the displacement.)

Comment. By measuring y as the displacement from equilibrium, it doesn't matter whether the mass is attached horizontally or vertically (gravity is taken into account by the extra stretch in the spring due to the mass).

Solving this DE, we find that the general solution is

$$y(t) = A\cos(\omega t) + B\sin(\omega t)$$

where $\omega = \sqrt{k/m}$ (note that the characteristic roots are $\pm i\sqrt{\frac{k}{m}}$). We observe that:

- The motion $y(t)$ is periodic with **period** $2\pi/\omega$.

  This follows from the fact that both $\cos(t)$ and $\sin(t)$ have period 2π.

- The **amplitude** of the motion $y(t)$ is $\sqrt{A^2 + B^2}$.

  This follows from the fact that $y(t) = A\cos(\omega t) + B\sin(\omega t) = r\cos(\omega t - \alpha)$ where (r, α) are the **polar coordinates** for (A, B). In particular, the amplitude is $r = \sqrt{A^2 + B^2}$.

  Can you explain the reason for being able to write $y(t)$ as $r\cos(\omega t - \alpha)$ using DEs? More on this next time…

Example 100. What is the period and the amplitude of the oscillations $\cos(4t) - 3\sin(4t)$?

Solution. The period is $\frac{2\pi}{4} = \frac{\pi}{2}$.

The amplitude is $\sqrt{1^2 + (-3)^2} = \sqrt{10}$.

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