**Example 90.** Consider the DE $y'' + 4y' + 4y = 2e^{3x} - 5e^{-2x}$.

(a) What is the simplest form (with undetermined coefficients) of a particular solution?

(b) Determine a particular solution using our results from Examples 88 and 89.

(c) Determine the general solution.

**Solution.**

(a) The "old" roots are $-2, -2$. The "new" roots are $3, -2$. Hence, there has to be a particular solution of the form $y_p = Ae^{3x} + Bx^2e^{-2x}$.

To find the (unique) values of $A$ and $B$, we can plug into the DE. Alternatively, we can break the problem into two pieces as illustrated in the next part.

(b) Write the DE as $Ly = 2e^{3x} - 5e^{-2x}$ where $L = D^2 + 4D + 4$. In Example 88 we found that $y_1 = \frac{1}{25}e^{3x}$ satisfies $Ly_1 = e^{3x}$. Also, in Example 89 we found that $y_2 = \frac{7}{2}x^2e^{-2x}$ satisfies $Ly_2 = 7e^{-2x}$.

By linearity, it follows that $L(Ay_1 + By_2) = ALy_1 + BLy_2 = Ae^{3x} + 7Be^{-2x}$.

To get a particular solution $y_p$ of our DE, we need $A = 2$ and $7B = -5$.

Hence, $y_p = 2y_1 - \frac{5}{7}y_2 = \frac{2}{25}e^{3x} - \frac{5}{2}x^2e^{-2x}$.

**Comment.** Of course, if we hadn't previously solved Examples 88 and 89, we could have plugged the result from the first part into the DE to determine the coefficients $A$ and $B$. On the other hand, breaking the inhomogeneous part ($2e^{3x} - 5e^{-2x}$) up into pieces (here, $e^{3x}$ and $e^{-2x}$) can help keep things organized, especially when working by hand.

(c) The general solution is $\frac{2}{25}e^{3x} - \frac{5}{2}x^2e^{-2x} + (C_1 + C_2x)e^{2x}$.

**Example 91.** Consider the DE $y'' - 2y' + y = 5\sin(3x)$.

(a) What is the simplest form (with undetermined coefficients) of a particular solution?

(b) Determine a particular solution.

(c) Determine the general solution.

**Solution.**

(a) Since $D^2 - 2D + 1 = (D-1)^2$, the "old" roots are $1, 1$. The "new" roots are $\pm 3i$. Hence, there has to be a particular solution of the form $y_p = A\cos(3x) + B\sin(3x)$.

(b) To find the values of $A$ and $B$, we plug into the DE.

$y_p' = -3A\sin(3x) + 3B\cos(3x)$

$y_p'' = -9A\cos(3x) - 9B\sin(3x)$

$y_p'' - 2y_p' + y_p = (-8A - 6B)\cos(3x) + (6A - 8B)\sin(3x) \overset{!}{=} 5\sin(3x)$

Equating the coefficients of $\cos(x)$, $\sin(x)$, we obtain the two equations $-8A - 6B = 0$ and $6A - 8B = 5$.

Solving these, we find $A = \frac{3}{10}$, $B = -\frac{2}{5}$. Accordingly, a particular solution is $y_p = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x)$.

(c) The general solution is $y(x) = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x) + (C_1 + C_2x)e^x$.

**Example 92.** Consider the DE $y'' - 2y' + y = 5e^{2x}\sin(3x) + 7xe^x$. What is the simplest form (with undetermined coefficients) of a particular solution?

**Solution.** Since $D^2 - 2D + 1 = (D-1)^2$, the "old" roots are $1, 1$. The "new" roots are $2 \pm 3i, 1, 1$. Hence, there has to be a particular solution of the form $y_p = Ae^{2x}\cos(3x) + Be^{2x}\sin(3x) + Cx^2e^x + Dx^3e^x$.

(We can then plug into the DE to determine the (unique) values of the coefficients $A, B, C, D$.)

**Example 93. (homework)** What is the shape of a particular solution of $y'' + 4y' + 4y = x\cos(x)$?

**Solution.** The "old" roots are $-2, -2$. The "new" roots are $\pm i, \pm i$. Hence, there has to be a particular solution of the form $y_p = (C_1 + C_2 x)\cos(x) + (C_3 + C_4 x)\sin(x)$.

**Continuing to find a particular solution.** To find the value of the $C_j$'s, we plug into the DE.

$y_p' = (C_2 + C_3 + C_4 x)\cos(x) + (C_4 - C_1 - C_2 x)\sin(x)$

$y_p'' = (2C_4 - C_1 - C_2 x)\cos(x) + (-2C_2 - C_3 - C_4 x)\sin(x)$

$$y_p'' + 4y_p' + 4y_p = (3C_1 + 4C_2 + 4C_3 + 2C_4 + (3C_2 + 4C_4)x)\cos(x) + (-4C_1 - 2C_2 + 3C_3 + 4C_4 + (-4C_2 + 3C_4)x)\sin(x) \overset{!}{=} x\cos(x).$$

Equating the coefficients of $\cos(x)$, $x\cos(x)$, $\sin(x)$, $x\sin(x)$, we get the equations $3C_1 + 4C_2 + 4C_3 + 2C_4 = 0$, $3C_2 + 4C_4 = 1$, $-4C_1 - 2C_2 + 3C_3 + 4C_4 = 0$, $-4C_2 + 3C_4 = 0$.

Solving (this is tedious!), we find $C_1 = -\frac{4}{125}$, $C_2 = \frac{3}{25}$, $C_3 = -\frac{22}{125}$, $C_4 = \frac{4}{25}$.

Hence, $y_p = \left(-\frac{4}{125} + \frac{3}{25}x\right)\cos(x) + \left(-\frac{22}{125} + \frac{4}{25}x\right)\sin(x)$.

**Example 94. (homework)** What is the shape of a particular solution of $y'' + 4y' + 4y = 4e^{3x}\sin(2x) - x\sin(x)$.

**Solution.** The "old" roots are $-2, -2$. The "new" roots are $3 \pm 2i, \pm i, \pm i$.

Hence, there has to be a particular solution of the form

$y_p = C_1 e^{3x}\cos(2x) + C_2 e^{3x}\sin(2x) + (C_3 + C_4 x)\cos(x) + (C_5 + C_6 x)\sin(x)$.

**Continuing to find a particular solution.** To find the values of $C_1, \ldots, C_6$, we plug into the DE. But this final step is so boring that we don't go through it here. Computers (currently?) cannot afford to be as selective; mine obediently calculated: $y_p = -\frac{4}{841}e^{3x}(20\cos(2x) - 21\sin(2x)) + \frac{1}{125}((-22 + 20x)\cos(x) + (4 - 15x)\sin(x))$

## A more general method for finding particular solutions: variation of parameters

The method of undetermined coefficients allows us to solve an inhomogeneous linear DE $Ly = f(x)$ for certain functions $f(x)$. The next method has no restriction on the functions $f(x)$ we can handle. The price to pay for this is that the method is usually more laborious.

**Review.** To find the general solution of an inhomogeneous linear DE $Ly = f(x)$, we only need to find a single **particular solution** $y_p$. Then the general solution is $y_p + y_h$, where $y_h$ is the general solution of $Ly = 0$.

**Theorem 95. (variation of parameters)** A particular solution to the inhomogeneous second-order linear DE $Ly = y'' + P_1(x)y' + P_0(x)y = f(x)$ is given by:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x), \quad u_1(x) = -\int \frac{y_2(x)f(x)}{W(x)}\mathrm{d}x, \quad u_2(x) = \int \frac{y_1(x)f(x)}{W(x)}\mathrm{d}x,$$

where $y_1, y_2$ are independent solutions of $Ly = 0$ and $W = y_1 y_2' - y_1' y_2$ is their Wronskian.

**Comment.** We obtain the general solution if we consider all possible constants of integration in the formula for $y_p$.

Armin Straub
straub@southalabama.edu

**Proof.** Let us look for a particular solution of the form $y_p = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$.

This "ansatz" is called **variation of constants/parameters**. We plug into the DE to determine conditions on $u_1$, $u_2$ so that $y_p$ is a solution. The DE will give us one condition and (since there are two unknowns), it is reasonable to expect that we can impose a second condition (labelled below as "our wish") to make our life simpler.

We compute $y_p' = \underbrace{u_1' y_1 + u_2' y_2}_{=0 \text{ (our wish)}} + u_1 y_1' + u_2 y_2'$ and, thus, $y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$.

["Our wish" was chosen so that $y_p''$ would only involve first derivatives of $u_1$ and $u_2$.]

Therefore, plugging into the DE results in

$$L y_p = u_1' y_1' + u_2' y_2' + \underbrace{u_1 y_1'' + u_2 y_2'' + P_1(x)(u_1 y_1' + u_2 y_2') + P_0(x)(u_1 y_1 + u_2 y_2)}_{= u_1 L y_1 + u_2 L y_2 = 0} \overset{!}{=} f(x).$$

We conclude that $y_p$ solves the DE if the following two conditions (the first is "our wish") are satisfied:

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0, \\ u_1' y_1' + u_2' y_2' &= f(x). \end{aligned}$$

These are linear equations in $u_1'$ and $u_2'$. Solving gives $u_1' = \frac{-y_2\, f(x)}{y_1 y_2' - y_1'\, y_2}$ and $u_2' = \frac{y_1\, f(x)}{y_1 y_2' - y_1'\, y_2}$, and it only remains to integrate. □

**Comment.** In matrix-vector form, the equations are $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix}$.

Our solution then follows from multiplying $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} = \frac{1}{y_1 y_2' - y_1'\, y_2} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}$ with $\begin{bmatrix} 0 \\ f(x) \end{bmatrix}$.

**Advanced comment.** $W = y_1 y_2' - y_1'\, y_2$ is called the **Wronskian** of $y_1$ and $y_2$. In general, given a linear homogeneous DE of order $n$ with solutions $y_1, \ldots, y_n$, the Wronskian of $y_1, \ldots, y_n$ is the determinant of the matrix where each column consists of the derivatives of one of the $y_i$. One useful property of the Wronskian is that it is nonzero if and only if the $y_1, \ldots, y_n$ are linearly independent and therefore generate the general solution.

## Example 96. Determine the general solution of $y'' - 2y' + y = \frac{e^x}{x}$.

**Solution.** This DE is of the form $L y = f(x)$ with $L = D^2 - 2D + 1$ and $f(x) = \frac{e^x}{x}$.

Since $L = (D-1)^2$, the homogeneous DE has the two solutions $y_1 = e^x$, $y_2 = x e^x$.

The corresponding Wronskian is $W = y_1 y_2' - y_1'\, y_2 = e^x (1+x) e^x - e^x (x e^x) = e^{2x}$.

By variation of parameters (Theorem 95), we find that a particular solution is

$$y_p = -y_1 \int \frac{y_2 f}{W} \mathrm{d}x + y_2 \int \frac{y_1 f}{W} \mathrm{d}x = -e^x \int 1 \mathrm{d}x + x e^x \int \frac{1}{x} \mathrm{d}x = x e^x (\ln|x| - 1).$$

The general solution therefore is $x e^x (\ln|x| - 1) + (C_1 + C_2 x) e^x$.

If we prefer, a simplified particular solution is $x e^x \ln|x|$ (because we can add any multiple of $x e^x$ to $y_p$). Then the general solution takes the simplified form $x e^x \ln|x| + (C_1 + C_2 x) e^x$.

**Comment.** Adding constants of integration in the formula for $y_p$, we get $-e^x (x + D_1) + x e^x (\ln|x| + D_2)$, which is the general solution. Any choice of constants suffices to give us a particular solution.

**Important comment.** Note that we cannot use the method of undetermined coefficients here because the inhomogeneous term $f(x) = \frac{e^x}{x}$ is not of the appropriate form. See the next example for a case where both methods can be applied.

**Example 97. (homework)** Determine the general solution of $y'' + 4y' + 4y = e^{3x}$.

(a) Using the method of undetermined coefficients.

(b) Using variation of constants.

**Solution.**

(a) We already did this in Example 88: The "old" roots are $-2, -2$. The "new" roots are $3$. Hence, there has to be a particular solution of the form $y_p = Ce^{3x}$. To find the value of $C$, we plug into the DE.

$y_p'' + 4y_p' + 4y_p = (9 + 4 \cdot 3 + 4)Ce^{3x} \overset{!}{=} e^{3x}$. Hence, $C = 1/25$.
Therefore, the general solution is $y(x) = \frac{1}{25}e^{3x} + (C_1 + C_2 x)e^{-2x}$.

(b) This DE is of the form $Ly = f(x)$ with $L = D^2 + 4D + 4$ and $f(x) = e^{3x}$.
Since $L = (D+2)^2$, the homogeneous DE has the two solutions $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$.
The corresponding Wronskian is $W = y_1 y_2' - y_1' y_2 = e^{-2x}(1-2x)e^{-2x} - (-2e^{-2x})xe^{-2x} = e^{-4x}$.
By variation of parameters (Theorem 95), we find that a particular solution is

$$\begin{aligned} y_p &= -y_1 \int \frac{y_2 f}{W} \mathrm{d}x + y_2 \int \frac{y_1 f}{W} \mathrm{d}x \\ &= -e^{-2x} \underbrace{\int xe^{5x} \mathrm{d}x}_{= \frac{1}{5}xe^{5x} - \frac{1}{25}e^{5x}} + xe^{-2x} \underbrace{\int e^{5x} \mathrm{d}x}_{= \frac{1}{5}e^{5x}} = \frac{1}{25}e^{3x}. \end{aligned}$$

The general solution therefore is $\frac{1}{25}e^{3x} + (C_1 + C_2 x)e^{-2x}$, which matches what we got before.

**Example 98. (homework)** Determine the general solution of $y'' + 4y' + 4y = 7e^{-2x}$.

(a) Using the method of undetermined coefficients.

(b) Using variation of constants.

**Solution.**

(a) We already did this in Example 89: The "old" roots are $-2, -2$. The "new" roots are $-2$. Hence, there has to be a particular solution of the form $y_p = Cx^2 e^{-2x}$. To find the value of $C$, we plug into the DE.
$y_p' = C(-2x^2 + 2x)e^{-2x}$
$y_p'' = C(4x^2 - 8x + 2)e^{-2x}$
$y_p'' + 4y_p' + 4y_p = 2Ce^{-2x} \overset{!}{=} 7e^{-2x}$
It follows that $C = 7/2$, so that $y_p = \frac{7}{2}x^2 e^{-2x}$. The general solution is $y(x) = \left(C_1 + C_2 x + \frac{7}{2}x^2\right)e^{-2x}$.

(b) This DE is of the form $Ly = f(x)$ with $L = D^2 + 4D + 4$ and $f(x) = 7e^{-2x}$.
Since $L = (D+2)^2$, the homogeneous DE has the two solutions $y_1 = e^{-2x}$, $y_2 = xe^{-2x}$.
The corresponding Wronskian is $W = y_1 y_2' - y_1' y_2 = e^{-2x}(1-2x)e^{-2x} - (-2e^{-2x})xe^{-2x} = e^{-4x}$.
By variation of parameters (Theorem 95), we find that a particular solution is

$$\begin{aligned} y_p &= -y_1 \int \frac{y_2 f}{W} \mathrm{d}x + y_2 \int \frac{y_1 f}{W} \mathrm{d}x \\ &= -e^{-2x} \underbrace{\int 7x \mathrm{d}x}_{= \frac{7}{2}x^2} + xe^{-2x} \underbrace{\int 7 \mathrm{d}x}_{= 7x} = \frac{7}{2}x^2 e^{-2x}. \end{aligned}$$

The general solution therefore is $\frac{7}{2}x^2 e^{-2x} + (C_1 + C_2 x)e^{-2x}$, which matches what we got before.