Inhomogeneous linear DEs: The method of undetermined coefficients

The method of undetermined coefficients allows us to solve any inhomogeneous linear DE $Ly = f(x)$ with constant coefficients if $f(x)$ is a polynomial times an exponential (or a linear combination of such terms).

More precisely, $Q(x)$ needs to be a solution of a homogeneous linear DE with constant coefficients.

Example 85. Determine the general solution of $y'' + 4y = 12x$.

Solution. The DE is $p(D)y = 12x$ with $p(D) = D^2 + 4$, which has roots $\pm 2i$. Thus, the general solution is $y(x) = y_p(x) + C_1\cos(2x) + C_2\sin(2x)$. It remains to find a particular solution y_p.

Since $D^2 \cdot (12x) = 0$, we apply D^2 to both sides of the DE to get the **homogeneous** DE $D^2(D^2+4)\cdot y = 0$.

Its general solution is $C_1 + C_2 x + C_3\cos(2x) + C_4\sin(2x)$ and y_p must be of this form. Indeed, there must be a particular solution of the simpler form $y_p = C_1 + C_2 x$ (because $C_3\cos(2x) + C_4\sin(2x)$ can be added to any y_p).

It remains to find appropriate values C_1, C_2 such that $y_p'' + 4y_p = 12x$. Since $y_p'' + 4y_p = 4C_1 + 4C_2 x$, comparing coefficients yields $4C_1 = 0$ and $4C_2 = 12$, so that $C_1 = 0$ and $C_2 = 3$. In other words, $y_p = 3x$.

Therefore, the general solution to the original DE is $y(x) = 3x + C_1\cos(2x) + C_2\sin(2x)$.

Example 86. Determine the general solution of $y'' + 4y' + 4y = e^{3x}$.

Solution. The DE is $p(D)y = e^{3x}$ with $p(D) = D^2 + 4D + 4 = (D+2)^2$, which has roots $-2, -2$. Thus, the general solution is $y(x) = y_p(x) + (C_1 + C_2 x)e^{-2x}$. It remains to find a particular solution y_p.

Since $(D-3)e^{3x} = 0$, we apply $(D-3)$ to the DE to get the **homogeneous** DE $(D-3)(D+2)^2 y = 0$.

Its general solution is $(C_1 + C_2 x)e^{-2x} + C_3 e^{3x}$ and y_p must be of this form. Indeed, there must be a particular solution of the simpler form $y_p = Ce^{3x}$.

To determine the value of C, we plug into the original DE: $y_p'' + 4y_p' + 4y_p = (9 + 4\cdot 3 + 4)Ce^{3x} \stackrel{!}{=} e^{3x}$. Hence, $C = 1/25$. Therefore, the general solution to the original DE is $y(x) = (C_1 + C_2 x)e^{-2x} + \frac{1}{25}e^{3x}$.

We found a recipe for solving nonhomogeneous linear DEs with constant coefficients.

Our approach works for $p(D)y = f(x)$ whenever the right-hand side $f(x)$ is the solution of some homogeneous linear DE with constant coefficients: $q(D)f(x) = 0$

Theorem 87. (method of undetermined coefficients) To find a particular solution y_p to an inhomogeneous linear DE with constant coefficients $p(D)y = f(x)$:

- Find $q(D)$ so that $q(D)f(x) = 0$. [This does not work for all $f(x)$.]
- It follows that y_p solves the **homogeneous** DE $q(D)\,p(D)y = 0$.

  The characteristic polynomial of this DE has roots:

  - The roots $r_1, \ldots, r_n$ of the polynomial $p(D)$ (the "old" roots).
  - The roots $s_1, \ldots, s_m$ of the polynomial $q(D)$ (the "new" roots).
- Let $y_1^{\text{new}}, \ldots, y_m^{\text{new}}$ be the "new" solutions (i.e. not solutions of the "old" $p(D)y = 0$).

  We plug into $p(D)y_p = f(x)$ to find (unique) C_i so that $y_p = C_1 y_1^{\text{new}} + \ldots + C_m y_m^{\text{new}}$.

Because of the final step, this approach is often called **method of undetermined coefficients**.

For which $f(x)$ does this work? By Theorem 70, we know exactly which $f(x)$ are solutions to homogeneous linear DEs with constant coefficients: these are linear combinations of exponentials $x^j e^{rx}$ (which includes $x^j e^{ax}\cos(bx)$ and $x^j e^{ax}\sin(bx)$).

Armin Straub
straub@southalabama.edu

Example 88. (again) Determine the general solution of $y'' + 4y' + 4y = e^{3x}$.

Solution. The "old" roots are $-2, -2$. The "new" roots are 3. Hence, there has to be a particular solution of the form $y_p = Ce^{3x}$. To find the value of C, we plug into the DE.

$y_p'' + 4y_p' + 4y_p = (9 + 4\cdot 3 + 4)Ce^{3x} \stackrel{!}{=} e^{3x}$. Hence, $C = 1/25$.

Therefore, the general solution is $y(x) = \frac{1}{25}e^{3x} + (C_1 + C_2 x)e^{-2x}$.

Example 89. Determine the general solution of $y'' + 4y' + 4y = 7e^{-2x}$.

Solution. The "old" roots are $-2, -2$. The "new" roots are -2. Hence, there has to be a particular solution of the form $y_p = Cx^2 e^{-2x}$. To find the value of C, we plug into the DE.

$y_p' = C(-2x^2 + 2x)e^{-2x}$

$y_p'' = C(4x^2 - 8x + 2)e^{-2x}$

$y_p'' + 4y_p' + 4y_p = 2Ce^{-2x} \stackrel{!}{=} 7e^{-2x}$

It follows that $C = 7/2$, so that $y_p = \frac{7}{2}x^2 e^{-2x}$. The general solution is $y(x) = \left(C_1 + C_2 x + \frac{7}{2}x^2\right)e^{-2x}$.