Review. A linear DE of order n is of the form

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x) y' + P_0(x) y = f(x).$$

The **general solution of linear DE** always takes the form

$$y(x) = y_p(x) + C_1 y_1(x) + \ldots + C_n\, y_n(x),$$

where y_p is any solution (called a **particular solution**) and $y_1,\ y_2,\ \ldots,\ y_n$ are solutions to the corresponding **homogeneous** linear DE.

- In terms of $D = \frac{\mathrm{d}}{\mathrm{d}x}$, the DE becomes: $Ly = f(x)$ with $L = D^n + P_{n-1}(x) D^{n-1} + \ldots + P_1(x) D + P_0(x)$.
- The inclusion of the $f(x)$ term makes $Ly = f(x)$ an **inhomogeneous** linear DE. The corresponding **homogeneous** DE is $Ly = 0$ (note that the zero function $y(x) = 0$ is a solution of $Ly = 0$).
- L is called a **linear differential operator**.
  - $L(C_1 y_1 + C_1 y_2) = C_1 L y_1 + C_2 L y_2$ **(linearity)**
    Comment. If you are familiar with linear algebra, think of L replaced with a matrix A and y_1, y_2 replaced with vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$. In that case, the same linearity property holds.
  - So, if y_1 solves $Ly = f(x)$, and y_2 solves $Ly = g(x)$, then $C_1 y_1 + C_2 y_2$ solves $C_1 f(x) + C_2\, g(x)$.
  - In particular, if y_1 and y_2 solve the homogeneous DE, then so does any linear combination $C_1 y_1 + C_2 y_2$. This explains why, for any homogeneous linear DE of order n, there are n solutions $y_1,\ y_2,\ \ldots,\ y_n$ such that the general solution is $y(x) = C_1 y_1(x) + \ldots + C_n\ y_n(x)$. Moreover, in that case, if we have a **particular solution** y_p of the inhomogeneous DE $Ly = f(x)$, then $y_p + C_1 y_1 + \ldots + C_n\, y_n$ is the general solution of $Ly = f(x)$.

Example 82. (review) Find the general solution of $y''' + 2y'' + y' = 0$.

Solution. The characteristic polynomial $p(D) = D(D+1)^2$ has roots $0, 1, 1$.
Hence, the general solution is $A + (B + Cx) e^x$.

Example 83. (review) Find the general solution of $y^{(7)} + 8y^{(6)} + 42y^{(5)} + 104y^{(4)} + 169y''' = 0$.

Use the fact that $-2 + 3i$ is a repeated characteristic root.

Solution. The characteristic polynomial $p(D) = D^3(D^2 + 4D + 13)^2$ has roots $0, 0, 0, -2 \pm 3i, -2 \pm 3i$.
[Since $-2 + 3i$ is a root so must be $-2 - 3i$. Repeating them once, together with $0, 0, 0$ results in 7 roots.]
Hence, the general solution is $(A + Bx + Cx^2) + (D + Ex)\, e^{-2x} \cos(3x) + (F + Gx) e^{-2x} \sin(3x)$.

Example 84. (preview) Determine the general solution of $y'' + 4y = 12x$. *Hint*: $3x$ is a solution.

Solution. Here, $p(D) = D^2 + 4$. Because of the hint, we know that a particular solution is $y_p = 3x$.
The homogeneous DE $p(D) y = 0$ has solutions $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$. [Make sure this is clear!]
Therefore, the general solution to the original DE is $y_p + C_1\, y_1 + C_2 y_2 = 3x + C_1 \cos(2x) + C_2 \sin(2x)$.

Just to make sure. The DE in operator notation is $Ly = f(x)$ with $L = D^2 + 4$ and $f(x) = 12x$.
Next. How to find the particular solution $y_p = 3x$ ourselves.

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