Notes for Lecture 19

Wed, 10/11/2023

Review. A homogeneous linear DE with constant coefficients is of the form $p(D)y=0$, where $p(D)$ is the characteristic polynomial polynomial. For each characteristic root r of multiplicity k, we get the k solutions $x^j e^{rx}$ for $j=0,1,\ldots,k-1$.

Example 73. Determine the general solution of $y'''-3y''+3y'-y=0$.

Solution. The characteristic polynomial $p(D)=D^3-3D^2+3D-1=(D-1)^3$ has roots $1,1,1$.
By Theorem 70, the general solution is $y(x)=(C_1+C_2x+C_3x^2)e^x$.

Example 74. Determine the general solution of $y'''-y''-5y'-3y=0$.

Solution. The characteristic polynomial $p(D)=D^3-D^2-5D-3=(D-3)(D+1)^2$ has roots $3,-1,-1$.
Hence, the general solution is $y(x)=C_1e^{3x}+(C_2+C_3x)e^{-x}$.

Example 75. (homework) Solve the IVP $y'''=8y''-16y'$ with $y(0)=1$, $y'(0)=4$, $y''(0)=0$.

Solution. The characteristic polynomial $p(D)=D^3-8D^2+16D=D(D-4)^2$ has roots $0,4,4$.
By Theorem 70, the general solution is $y(x)=C_1+(C_2+C_3x)e^{4x}$.
Using $y'(x)=(4C_2+C_3+4C_3x)e^{4x}$ and $y''(x)=4(4C_2+2C_3+4C_3x)e^{4x}$, the initial conditions result in the equations $C_1+C_2=1$, $4C_2+C_3=4$, $16C_2+8C_3=0$.
Solving these (start with the last two equations) we find $C_1=-1$, $C_2=2$, $C_3=-4$.
Hence the unique solution to the IVP is $y(x)=-1+(2-4x)e^{4x}$.
Important comment. Check that $y(x)$ indeed solves the IVP.

Example 76. Determine the general solution of $y^{(6)}=3y^{(5)}-4y'''$.

Solution. This DE is of the form $p(D)\,y=0$ with $p(D)=D^6-3D^5+4D^3=D^3(D-2)^2(D+1)$.
The characteristic roots are $2,2,0,0,0,-1$.
By Theorem 70, the general solution is $y(x)=(C_1+C_2x)e^{2x}+C_3+C_4x+C_5x^2+C_6e^{-x}$.

Example 77. Consider the function $y(x)=3xe^{-2x}+7e^x$. Determine a homogeneous linear DE with constant coefficients of which $y(x)$ is a solution.

Solution. In order for $y(x)$ to be a solution of $p(D)y=0$, the characteristic roots must include $-2,-2,1$.
The simplest choice for $p(D)$ thus is $p(D)=(D+2)^2(D-1)=D^3+3D^2-4$.
Accordingly, $y(x)$ is a solution of $y'''+3y''-4y=0$.

Example 78. Consider the function $y(x)=3xe^{-2x}+7$. Determine a homogeneous linear DE with constant coefficients of which $y(x)$ is a solution.

Solution. In order for $y(x)$ to be a solution of $p(D)y=0$, the characteristic roots must include $-2,-2,0$.
The simplest choice for $p(D)$ thus is $p(D)=(D+2)^2D=D^3+4D^2+4D$.
Accordingly, $y(x)$ is a solution of $y'''+4y''+4y'=0$.

Armin Straub
straub@southalabama.edu

Real form of complex solutions

Let's recall some basic facts about **complex numbers**:

- Every complex number can be written as $z = x + iy$ with real x, y.
- Here, the imaginary unit i is characterized by solving $x^2 = -1$.

  Important observation. The same equation is solved by $-i$. This means that, algebraically, we cannot distinguish between $+i$ and $-i$.

- The **conjugate** of $z = x + iy$ is $\bar{z} = x - iy$.

  Important comment. Since we cannot algebraically distinguish between $\pm i$, we also cannot distinguish between z and $\bar{z}$. That's the reason why, in problems involving only real numbers, if a complex number $z = x + iy$ shows up, then its **conjugate** $\bar{z} = x - iy$ has to show up in the same manner. With that in mind, have another look at the examples below.

- The **real part** of $z = x + iy$ is x and we write $\operatorname{Re}(z) = x$.

  Likewise the **imaginary part** is $\operatorname{Im}(z) = y$.

  Observe that $\operatorname{Re}(z) = \frac{1}{2}(z + \bar{z})$ as well as $\operatorname{Im}(z) = \frac{1}{2i}(z - \bar{z})$.

Theorem 79. (Euler's identity) $e^{ix} = \cos(x) + i\sin(x)$

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.
[Check that by computing the derivatives and verifying the initial condition! As we did in class.] □

Comment. It follows that $\cos(x) = \operatorname{Re}(e^{ix}) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \operatorname{Im}(e^{ix}) = \frac{1}{2i}(e^{ix} - e^{-ix})$.

Example 80. Determine the general solution of $y'' + y = 0$.

Solution. (complex numbers in general solution) The characteristic polynomial is $D^2 + 1$ which has no roots over the reals. Over the **complex numbers**, by definition, the roots are i and $-i$.
So the general solution is $y(x) = C_1 e^{ix} + C_2 e^{-ix}$.

Solution. (real general solution) On the other hand, we easily check that $y_1 = \cos(x)$ and $y_2 = \sin(x)$ are two solutions. Hence, the general solution can also be written as $y(x) = D_1\cos(x) + D_2\sin(x)$.

Important comment. That we have these two different representations is a consequence of Euler's identity (Theorem 79) by which $e^{\pm ix} = \cos(x) \pm i\sin(x)$.
On the other hand, $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$.
[Recall that the first formula is an instance of $\operatorname{Re}(z) = \frac{1}{2}(z + \bar{z})$ and the second of $\operatorname{Im}(z) = \frac{1}{2i}(z - \bar{z})$.]

Example 81. Determine the general solution of $y'' - 4y' + 13y = 0$ using only real numbers.

Solution. The characteristic polynomial $p(D) = D^2 - 4D + 13$ has roots $2 + 3i, 2 - 3i$.

[We can use the quadratic formula to find these roots as $\frac{4 \pm \sqrt{4^2 - 4\cdot 13}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$.]

Hence, the general solution in real form is $y(x) = C_1 e^{2x}\cos(3x) + C_2 e^{2x}\sin(3x)$.
Note. $e^{(2\pm 3i)x} = e^{2x}e^{\pm 3ix} = e^{2x}(\cos(3x) \pm i\sin(3x))$

Armin Straub
straub@southalabama.edu