Numerically "solving" DEs: Euler's method

Recall that the general form of a first-order initial value problem is

$$y' = f(x, y), \quad y(x_0) = y_0.$$

Further recall that, under mild assumptions on $f(x, y)$, such an IVP has a unique solution $\psi(x)$.

Comment. While deriving Euler's method, we write $\psi(x)$ instead of $y(x)$ simply to not confuse ourselves (note that y is also used as the variable in the differential equation and y will also be used as the variable when writing down equations for tangent lines).

We have learned some techniques for (exactly) solving DEs. On the other hand, many DEs that arise in practice cannot be solved by these techniques (or more fancy ones).

Instead, it is common in practice to approximate the solution $\psi(x)$ to our IVP. Euler's method is the simplest example of how this can done. The key idea is to locally approximate $\psi(x)$ by tangent lines.

Example 54. Spell out the equation for the tangent line of a function $\psi(x)$ at the point (x_0, y_0).

Solution. $y = y_0 + \psi'(x_0)(x - x_0)$

Note that, in our case where $\psi(x)$ solves the IVP, we know what $\psi'(x_0)$ is! Namely,

$$\psi'(x_0) = f(x_0, \psi(x_0)) = f(x_0, y_0).$$

Comment. Make sure you see that the right-hand is a number that we can easily compute.

We therefore know the equation of the tangent line of $\psi(x)$ at the initial point (x_0, y_0). It is

$$y = y_0 + f(x_0, y_0)(x - x_0).$$

The idea is to use this tangent line as an approximation of $\psi(x)$ for just a little bit, namely from x_0 to $x_0 + h$ for a small **step size** h. We set $x_1 = x_0 + h$ and compute y_1 as the corresponding y-value on the tangent line using

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0) = \boxed{y_0 + h f(x_0, y_0)}.$$

Note that $y_1 \approx \psi(x_1)$ is a good approximation if h is sufficiently small.

At this point, we have gone from our initial point (x_0, y_0) to a next (approximate) point (x_1, y_1). We now repeat what we did to get a third point (x_2, y_2) with $x_2 = x_1 + h$. Continuing in this way, we obtain Euler's method:

(Euler's method) To approximate the solution $y(x)$ of the IVP $y' = f(x, y)$, $y(x_0) = y_0$, we start with the point (x_0, y_0) and a step size h. We then compute:

$$\begin{aligned} x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + h f(x_n, y_n) \end{aligned}$$

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Example 55. Consider the IVP $\frac{dy}{dx} = (2x - 3y)^2 + \frac{2}{3}$, $y(1) = \frac{1}{3}$.

(a) Approximate the solution $y(x)$ for $x \in [1, 2]$ using Euler's method with 2 steps.

(b) Approximate the solution $y(x)$ for $x \in [1, 2]$ using Euler's method with 3 steps.

(c) Solve this IVP exactly. Compare the values at $x = 2$.

Solution.

(a) The step size is $h = \frac{2-1}{2} = \frac{1}{2}$. We apply Euler's method with $f(x, y) = (2x - 3y)^2 + \frac{2}{3}$:

$$\begin{array}{ll} x_0 = 1 & y_0 = \frac{1}{3} \\ x_1 = \frac{3}{2} & y_1 = y_0 + h f(x_0, y_0) = \frac{1}{3} + \frac{1}{2} \cdot \left[\left(2 \cdot 1 - 3 \cdot \frac{1}{3}\right)^2 + \frac{2}{3}\right] = \frac{7}{6} \\ x_2 = 2 & y_2 = y_1 + h f(x_1, y_1) = \frac{7}{6} + \frac{1}{2} \cdot \frac{11}{12} = \frac{13}{8} \end{array}$$

In particular, the approximation for $y(2)$ is $y_2 = \frac{13}{8} = 1.625$.

(b) The step size is $h = \frac{2-1}{3} = \frac{1}{3}$. We again apply Euler's method with $f(x, y) = (2x - 3y)^2 + \frac{2}{3}$:

$$\begin{array}{ll} x_0 = 1 & y_0 = \frac{1}{3} \\ x_1 = \frac{4}{3} & y_1 = y_0 + h f(x_0, y_0) = \frac{1}{3} + \frac{1}{3} \cdot \left[\left(2 \cdot 1 - 3 \cdot \frac{1}{3}\right)^2 + \frac{2}{3}\right] = \frac{8}{9} \\ x_2 = \frac{5}{3} & y_2 = y_1 + h f(x_1, y_1) = \frac{8}{9} + \frac{1}{3} \cdot \frac{2}{3} = \frac{10}{9} \\ x_3 = 2 & y_3 = y_2 + h f(x_2, y_2) = \frac{10}{9} + \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{3} \end{array}$$

In particular, the approximation for $y(2)$ is $y_3 = \frac{4}{3} \approx 1.333$.

(c) We solved this IVP in Example 37 using the substitution $u = 2x - 3y$ followed by separation of variables. We found that the unique solution of the IVP is $y(x) = \frac{2}{3}x - \frac{1}{3(3x-2)}$.

In particular, the exact value at $x = 2$ is $y(2) = \frac{5}{4} = 1.25$.

We observe that our approximations for $y(2) = 1.25$ improved from 1.625 to 1.333 as we increased the number of steps (equivalently, we decreased the step size h from $\frac{1}{2}$ to $\frac{1}{3}$).

For comparison. With 10 steps (so that $h = \frac{1}{10}$), the approximation improves to $y(2) \approx 1.259$.

Example 56. Consider the IVP $y' = y$, $y(0) = 1$. Approximate the solution $y(x)$ for $x \in [0, 1]$ using Euler's method with 4 steps. In particular, what is the approximation for $y(1)$?

Comment. Of course, the real solution is $y(x) = e^x$. In particular, $y(1) = e \approx 2.71828$.

Solution. The step size is $h = \frac{1-0}{4} = \frac{1}{4}$. We apply Euler's method with $f(x, y) = y$:

$$\begin{array}{ll} x_0 = 0 & y_0 = 1 \\ x_1 = \frac{1}{4} & y_1 = y_0 + h f(x_0, y_0) = 1 + \frac{1}{4} \cdot 1 = \frac{5}{4} = 1.25 \\ x_2 = \frac{1}{2} & y_2 = y_1 + h f(x_1, y_1) = \frac{5}{4} + \frac{1}{4} \cdot \frac{5}{4} = \frac{5^2}{4^2} = 1.5625 \\ x_3 = \frac{3}{4} & y_3 = y_2 + h f(x_2, y_2) = \frac{5^2}{4^2} + \frac{1}{4} \cdot \frac{5^2}{4^2} = \frac{5^3}{4^3} \approx 1.9531 \\ x_4 = 1 & y_4 = y_3 + h f(x_3, y_3) = \frac{5^3}{4^3} + \frac{1}{4} \cdot \frac{5^3}{4^3} = \frac{5^4}{4^4} \approx 2.4414 \end{array}$$

In particular, the approximation for $y(1)$ is $y_4 \approx 2.4414$.

Comment. Can you see that, if instead we start with $h = \frac{1}{n}$, then we similarly get $x_i = \frac{(n+1)^i}{n^i}$ for $i = 0, 1, \ldots, n$? In particular, $y(1) \approx y_n = \frac{(n+1)^n}{n^n} = \left(1 + \frac{1}{n}\right)^n \to e$ as $n \to \infty$. Do you recall how to derive this final limit?

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Preview: Solving linear differential equations with constant coefficients

Let us have another look at Example 11. Note that the DE is a second-order linear differential equation with constant coefficients. Our upcoming goal will be to solve all such equations.

Example 57. Find the general solution to $y'' = y' + 6y$.

Solution. We look for solutions of the form e^{rx}.

Plugging e^{rx} into the DE, we get $r^2 e^{rx} = r e^{rx} + 6 e^{rx}$ which simplifies to $r^2 - r - 6 = 0$.

This is called the **characteristic equation**. Its solutions are $r = -2, 3$ (the **characteristic roots**).

This means we found the two solutions $y_1 = e^{-2x}$, $y_2 = e^{3x}$.

The general solution to the DE is $C_1 e^{-2x} + C_2 e^{3x}$.

Comment. In the final step, we used an important principle that is true for linear (!) homogeneous DEs. Namely, if we have solutions $y_1, y_2, \ldots$ then any linear combination $C_1 y_1 + C_2 y_2 + \ldots$ is a solution as well. We will discuss this soon but, for now, check that $C_1 e^{-2x} + C_2 e^{3x}$ is indeed a solution by plugging it into the DE.

Example 58. (extra) Find the general solution to $y''' = y'' + 6y'$.

Solution. We look for solutions of the form e^{rx}.

Plugging e^{rx} into the DE, we get $r^3 e^{rx} = r^2 e^{rx} + 6 r e^{rx}$ which simplifies to $r^3 - r^2 - 6r = r(r^2 - r - 6) = 0$.

As in Example 57, $r^2 - r - 6 = 0$ has the solutions $r = -2, 3$.

Overall, $r(r^2 - r - 6) = 0$ has the three solutions $-2, 3, 0$.

This means we found the three solutions $y_1 = e^{-2x}$, $y_2 = e^{3x}$, $y_3 = e^{0x} = 1$.

The general solution to the DE is $C_1 e^{-2x} + C_2 e^{3x} + C_3$.

Alternatively. We can substitute $u = y'$, in which case the new DE is $u'' = u' + 6u$. From Example 57, we know that $u = C_1 e^{-2x} + C_2 e^{3x}$.

Hence, the general solution of the initial DE is $y = \int u \, \mathrm{d}x = -\frac{1}{2} C_1 e^{-2x} + \frac{1}{3} C_2 e^{3x} + C$.

Note that we can set $D_1 = -\frac{1}{2} C_1$, $D_2 = \frac{1}{3} C_2$, $D_3 = C$ to write this as $D_1 e^{-2x} + D_2 e^{3x} + D_3$, which matches our earlier solution.

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