Acceleration–velocity models

To model a falling object, we let $y(t)$ be its height at time t.

Then physics has names for $y'(t)$ and $y''(t)$: these are the **velocity** and the **acceleration**.

Physics tells us that objects fall due to gravity (and that it makes already-falling objects fall faster; in other words, gravity accelerates falling objects). Physicists have measured that, on earth, the the gravitational acceleration is $g \approx 9.81\mathrm{m/s}^2$.

If we only take earth's gravitation into account, then the fall is therefore modeled by

$$y''(t) = -g.$$

Example 52. A ball is dropped from a 100m tall building. How long until it reaches the ground? What is the speed when it hits the ground?

Solution. Let $y(t)$ be the height (in meters) at which the ball is at time t (in seconds).

As above, physics tells us that an object falling due to gravity (and ignoring everything else) satisfies the DE $y'' = -g$ where $g \approx 9.81$. We further know the initial values $y(0) = 100$, $y'(0) = 0$.

Substituting $v = y'$ in the DE, we get $v' = -g$. This DE is solved by $v(t) = -gt + C$.

Hence, $y(t) = \int v(t)\mathrm{d}t = -\frac{1}{2}gt^2 + Ct + D$.

The initial conditions $y(0) = 100$, $y'(0) = 0$ tell us that $D = 100$ and $C = 0$.

Thus $y(t) = -\frac{1}{2}gt^2 + 100$.

The ball reaches the ground when $y(t) = -\frac{1}{2}gt^2 + 100 = 0$, that is after $t = \sqrt{200/g} \approx 4.52$ seconds.

The speed then is $|y'(4.52)| \approx 44.3\,\mathrm{m}/s$.

For many applications, one needs to take air resistance into account.

This is actually less well understood than one might think, and the physics quickly becomes rather complicated. Typically, air resistance is somewhere in between the following two cases:

- Under certain assumptions, physics suggests that air resistance is proportional to the square of the velocity.

  Comment. A simplistic way to think about this is to imagine the falling object to bump into (air) particles; if the object falls twice as fast, then the momentum of the particles it bumps into is twice as large and it bumps into twice as many of them.

- In other cases such as "relatively slowly" falling objects, one might empirically observe that air resistance is proportional to the velocity itself.

  Comment. One might imagine that, at slow speed, the falling object doesn't exactly bump into particles but instead just gently pushes them aside; so that at twice the speed it only needs to gently push twice as often.

Example 53. When modeling the (slow) fall of a parachute, physics suggests that the air resistance is roughly proportional to velocity. If $y(t)$ is the parachute's height at time t, then the corresponding DE is $y'' = -g - \rho y'$ where $\rho > 0$ is a constant.

Comment. Note that $-\rho y' > 0$ because $y' < 0$. Thus, as intended, air resistance is acting in the opposite direction as gravity and slowing down the fall.

Determine the general solution of the DE.

Solution. Substituting $v = y'$, the DE becomes $v' + \rho v = -g$.

This is a linear DE. To solve it, we determine that the integrating factor is $\exp(\int \rho \mathrm{d}t) = e^{\rho t}$.

Multiplying the DE with that factor and integrating, we obtain $e^{\rho t} v = \int -g e^{\rho t} \mathrm{d}t = -\frac{g}{\rho} e^{\rho t} + C$.

Hence, $v(t) = -\frac{g}{\rho} + C e^{-\rho t}$.

Correspondingly, the general solution of the DE is $y(t) = \int v(t) \mathrm{d}t = -\frac{g}{\rho} t - \frac{C}{\rho} e^{-\rho t} + D$.

Comment. Note that $\lim_{t \to \infty} v(t) = -\frac{g}{\rho}$. In other words, the **terminal velocity** is $v_\infty = -\frac{g}{\rho}$.

This is an interesting mathematical consequence of the DE. (And important for the idea behind a parachute!)

Note that, if we know that there is a terminal speed, then we can actually determine its value v_∞ from the DE without solving it by setting $v' = 0$ (because, once the terminal speed is reached, the velocity does not change anymore) in $v' + \rho v = -g$. This gives us $\rho v_\infty = -g$ and, hence, $v_\infty = -g / \rho$ as above.