Mixing problems

Example 51. A tank contains 20gal of pure water. It is filled with brine (containing 5lb/gal salt) at a rate of 3gal/min. At the same time, well-mixed solution flows out at a rate of 2gal/min. How much salt is in the tank after t minutes?

Solution.
(Part I. determining a DE) Let $x(t)$ denote the amount of salt (in lb) in the tank after time t (in min).

At time t, the concentration of salt (in lb/gal) in the tank is $\frac{x(t)}{V(t)}$ where $V(t) = 20 + (3-2)t = 20 + t$ is the volume (in gal) in the tank.

In the time interval $[t, t+\Delta t]$: $\Delta x \approx 3 \cdot 5 \cdot \Delta t - 2 \cdot \frac{x(t)}{V(t)} \cdot \Delta t$.

Hence, $x(t)$ solves the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = 15 - 2 \cdot \frac{x}{20+t}$ with $x(0) = 0$.

Comment. Can you explain why the equation for Δx is only approximate but why the final DE is exact? [Hint: $x(t)/V(t)$ is the concentration at time t but we are using it for Δx at other times as well.]

(Part II. solving the DE) Since this is a linear DE, we can solve it as follows:

- Write the DE in the standard form as $\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{2}{20+t}x = 15$.
- The integrating factor is $f(t) = \exp\left(\int \frac{2}{20+t}\mathrm{d}t\right) = \exp(2\ln|20+t|) = (20+t)^2$.
- Multiply the DE (in standard form) by $f(t) = (20+t)^2$ to get $\underbrace{(20+t)^2\frac{\mathrm{d}x}{\mathrm{d}t} + 2(20+t)x}_{=\frac{\mathrm{d}}{\mathrm{d}t}[(20+t)^2 x]} = 15(20+t)^2$.
- Integrate both sides to get $(20+t)^2 x = 15\int (20+t)^2 \mathrm{d}t = 5(20+t)^3 + C$.

Hence the general solution to the DE is $x(t) = 5(20+t) + \frac{C}{(20+t)^2}$. Using $x(0) = 0$, we find $C = -5 \cdot 20^3$.

We conclude that, after t minutes, the tank contains $x(t) = 5(20+t) - \frac{5 \cdot 20^3}{(20+t)^2}$ pounds of salt.

Comment. As a consequence, $x(t) \approx 5(20+t) = 5V(t)$ for large t. Can you explain why that makes perfect sense and why we could have predicted this from the very beginning (without deriving a DE and solving it)?

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