Notes for Lecture 12

Wed, 9/20/2023

Further population models

Let $P(t)$ be the size of the population that we wish to model at time t.

Denote with $\beta(t)$ and $\delta(t)$ the birth and death rate at time t, measured in number of births or deaths per unit of population per unit of time.

In the time interval $[t, t + \Delta t]$, we have that

$$\Delta P \approx \beta(t)P(t)\Delta t - \delta(t)P(t)\Delta t.$$

Comment. The reason that this is not an exact equation is that the rates $\beta(t)$ and $\delta(t)$ are allowed to change with t. In the above, we used these rates at time t for all times in $[t, t + \Delta t]$. This is a good approximation if Δt is small.

Divide both sides by Δt and let $\Delta t \to 0$ to obtain the general differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta(t) - \delta(t))P.$$

Given certain scenarios, we now make corresponding reasonable choices for $\beta(t)$ and $\delta(t)$.

- **(basic)** If the rates $\beta(t)$ and $\delta(t)$ are constant over time, the DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta - \delta)P$.
  This is the exponential model of population growth.

- **(limited supply)** If supply is limited, the birth rate will decrease as P increases. The simplest such relationship would be a linear dependence, which would take the form $\beta(t) = \beta_0 - \beta_1 P$.

  On the other hand, we still assume that $\delta(t)$ is constant. (However, depending on circumstances, it could also be reasonable to assume that $\delta(t)$ increases as P increases.)

  With these assumptions, the corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta_0 - \beta_1 P - \delta)P$.

  This is the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(1 - P/M)$ with $k = \beta_0 - \delta$ and $\frac{k}{M} = \beta_1$.

- **(rare isolated species)** If the population consists of rare and isolated specimen which rely on chance encounters to reproduce, then it is reasonable to assume that the birth rate $\beta(t)$ is proportional to $P(t)$ (larger $P(t)$ means more possibilities for chance encounters). Once more, we assume that $\delta(t)$ constant.

  With these assumptions, the corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (kP - \delta)P$.

  This is, again, the logistic equation.

- **(rare isolated species with very long life)** As before, for a rare isolated population, it is reasonable to assume that $\beta(t)$ is proportional to $P(t)$. If, in addition, our specimen have very long life, then we would assume that $\delta(t) = 0$.

  The corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP^2$. Solutions are $P(t) = \frac{1}{C - kt}$ where $P(0) = 1/C$. (Do it!)

  Comment. Note that $P(t) \to \infty$ as $t \to C/k$. This explosion (which implies population growth beyond exponential growth) emphasizes that we can only use the DE while our initial assumptions are satisfied. Here, the DE is no longer appropriate when our species is no longer rare because $P(t)$ is too large.

- **(spread of contagious incurable virus)** Let $P(t)$ count the number of infected population units among a (constant) total of N. Since the virus is incurable, we have $\delta(t) = 0$. On the other hand, it is reasonable to assume that $\beta(t)$ is proportional to $N - P$ (the number of people that can still be infected).

  The resulting DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(N - P)$. Once again, this is the logistic equation.

- **(harvesting)** Suppose that h population units are harvested each unit of time.

  Then the DE becomes $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta(t) - \delta(t))P - h$.

  For instance. $\frac{\mathrm{d}P}{\mathrm{d}t} = kP - h$ has the solution $P(t) = Ce^{kt} + h/k$. In that case, we get exponential growth if $C > 0$. Note that $P(0) = C + h/k$. In terms of the initial population $P(0)$, we therefore get exponential growth if $P(0) > h/k$. (Also see next example!)

Example 49. A biotech company is growing certain microorganisms in the lab. From experience they know that the growth (number of organisms per day) of the microorganisms is well modeled by an exponential model with proportionality constant $k=5$ (per day). What is the optimal rate (in number of organisms per day) at which the company can harvest the microorganisms?

Solution. (long version via solving the DE) Without harvesting, the growth is modeled by $\frac{\mathrm{d}P}{\mathrm{d}t}=5P$ (the exponential model). Here, P is the number of organisms and t measures time in days. (Always think about your units in applications!)

If harvesting occurs at the rate of h number of organisms per day, the population model needs to be adjusted to

$$\frac{\mathrm{d}P}{\mathrm{d}t}=5P-h.$$

Since h is a constant, we can solve this DE using separation of variables. Alternatively, the DE is linear and we can therefore solve it using an integrating factor. For practice, we do both:

- **(separation of variables)** Integrating $\frac{1}{5P-h}\mathrm{d}P=\mathrm{d}t$, we find $\frac{1}{5}\ln|5P-h|=t+C$, which we simplify to $|5P-h|=e^{5t+5C}$. It following that $5P-h=\pm e^{5t}e^{5C}=Be^{5t}$ where we wrote $B=\pm e^{5C}$ (note that the sign is fixed and cannot change).

  Thus, the general solution of the DE is $P(t)=\frac{h}{5}+Ae^{5t}$ (where we wrote $A=\frac{B}{5}$).

- **(integrating factor)** Since this is a linear DE, we can solve it as follows:
  - We write the DE in the form $\frac{\mathrm{d}P}{\mathrm{d}t}-5P=-h$.
  - The integrating factor is $f(t)=\exp(\int -5\,\mathrm{d}t)=e^{-5t}$.
  - Multiply the (rewritten) DE by $f(t)$ to get $\underbrace{e^{-5t}\frac{\mathrm{d}P}{\mathrm{d}t}-5e^{-5t}P}_{=\frac{\mathrm{d}}{\mathrm{d}t}[e^{-5t}P]}=-he^{-5t}$.
  - Integrate both sides to get $e^{-5t}P=-h\int e^{-5t}\mathrm{d}t=\frac{h}{5}e^{-5t}+C$.

  Hence the general solution to the DE is $P(t)=\frac{h}{5}+Ce^{5t}$.

In either case, we found that $P(t)=\frac{h}{5}+Ce^{5t}$. In order to be able to continually harvest, we need to make sure that $C\geqslant 0$. In terms of the initial population, we get $P(0)=\frac{h}{5}+C$ so that $C=P(0)-\frac{h}{5}$.

Thus the condition $C\geqslant 0$ becomes $P(0)-\frac{h}{5}\geqslant 0$ or, equivalently, $h\leqslant 5P(0)$. Thus, the optimal rate of harvesting is $h=5P(0)$.

Solution. (short version) As before, we observe that, if harvesting occurs at the rate of h number of organisms per day, then our population model is

$$\frac{\mathrm{d}P}{\mathrm{d}t}=5P-h.$$

In order to be able to continually harvest, we need to make sure that $\frac{\mathrm{d}P}{\mathrm{d}t}\geqslant 0$ (clearly, this is sufficient; we can also see that it is necessary since a decreased population should result in a lower optimal harvesting rate).

We thus get the condition $5P-h\geqslant 0$. Since the population is not decreasing (because $\frac{\mathrm{d}P}{\mathrm{d}t}\geqslant 0$), this is equivalent to $5P(0)-h\geqslant 0$ or, equivalently, $h\leqslant 5P(0)$. Again, we conclude that the optimal rate of harvesting is $h=5P(0)$.

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Review. The **logistic equation** is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$.

Here, k is the growth rate and M is the carrying capacity.

The general solution is $P(t) = \frac{M}{1 + Ce^{-kt}}$ where $C = \frac{M}{P(0)} - 1$.

Example 50. In a city with a fixed population N, the time rate of change of the number P of people who have heard a certain rumor is proportional to the product of P and $N - P$. Suppose initially 10% have heard the rumor and after a week this number has grown to 20%. What percentage will this number reach after one more week?

Solution. $\frac{\mathrm{d}P}{\mathrm{d}t} = \gamma P(N - P)$. $P(0) = 0.1N$ and $P(1) = 0.2N$. We need $P(2)$.

Note that this is a logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)$ with $k = \gamma N$ and carrying capacity N.

It therefore has the general solution $P(t) = \frac{N}{1 + Ce^{-kt}}$.

Using $P(0) = \frac{N}{1 + C} = 0.1N$, we find that $C = 9$.

Using $P(1) = \frac{N}{1 + 9e^{-k}} = 0.2N$, we further find that $e^{-k} = \frac{4}{9}$.

We could solve for k but note that it is more pleasing to use $e^{-kt} = (e^{-k})^t = \left(\frac{4}{9}\right)^t$ in our formula for $P(t)$.

We conclude that $P(t) = \frac{N}{1 + 9\left(\frac{4}{9}\right)^t}$.

In particular, $P(2) = \frac{N}{1 + 9 \cdot \frac{16}{81}} = \frac{9}{25}N$ which is 36%.