Notes for Lecture 11

Mon, 9/18/2023

The logistic model of population growth

If the population is constrained by resources, then $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ is not a good model. A model to take that into account is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$. This is the **logistic equation**.

M is called the carrying capacity:

- Note that if $P \ll M$ then $1 - \frac{P}{M} \approx 1$ and we are back to the simpler exponential model. This means that the population P will grow (nearly) exponentially if P is much less than the carrying capacity M.
- On the other hand, if $P > M$ then $1 - \frac{P}{M} < 0$ so that (assuming $k > 0$) $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$, which means that the population P is shrinking if it exceeds the carrying capacity M.

Comment. If $P(t)$ is the size of a population, then P'/P can be interpreted as its *per capita growth rate*.
Note that in the exponential model we have that $P'/P = k$ is constant.
On the other hand, in the logistic model we have that $P'/P = k(1 - P/M)$ is a linear function.

Example 45. Solve the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$.

Solution. This is a separable DE: $\frac{1}{P(1 - \frac{P}{M})}\mathrm{d}P = k\mathrm{d}t$.

To integrate the left-hand side, we use partial fractions: $\frac{1}{P(1 - \frac{P}{M})} = \frac{1}{P} + \frac{1/M}{1 - \frac{P}{M}} = \frac{1}{P} - \frac{1}{P - M}$.

After integrating, we obtain $\ln|P| - \ln|P - M| = kt + A$.

Equivalently, $\ln\left|\frac{P}{P - M}\right| = kt + A$ so that $\frac{P}{P - M} = \pm e^{kt + A} = Be^{kt}$ where $B = \pm e^A$.

Solving for P, we conclude that the general solution is

$$P(t) = \frac{BMe^{kt}}{Be^{kt} - 1} = \frac{M}{1 + Ce^{-kt}},$$

where we replaced the free parameter B with $C = -1/B$.

Initial population. Note that the initial population is $P(0) = \frac{M}{1 + C}$. Equivalently, $C = \frac{M}{P(0)} - 1$ which expresses the free parameter C in terms of the initial population.

Comment. Note that $B = \pm e^A$ can be any real number except 0. However, we can easily check that $B = 0$ also gives us a solution to the DE (namely, the trivial solution $P = 0$). This solution was "lost" when we divided by P to separate variables.

Exercise. Note that the logistic equation is a Bernoulli equation. As an alternative to separation of variables, we can therefore solve it by transforming it to a linear DE.

Review of partial fractions. Recall that partial fractions tells us that fractions like $\frac{p(x)}{(x - r_1)(x - r_2)\cdots}$ (with the numerator of smaller degree than the denominator; and with the r_j distinct) can be written as a sum of terms of the form $\frac{A_j}{x - r_j}$ for suitable constants A_j.

In our case, this tells us that $\frac{1}{P(1 - P/M)} = \frac{A}{P} + \frac{B}{1 - P/M}$ for certain constants A and B.

Multiply both sides by P and set $P = 0$ to find $A = 1$.

Multiply both sides by $1 - P/M$ and set $P = M$ to find $B = 1/M$. This is what we used above.

The **logistic equation** with growth rate k and carrying capacity M is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right).$$

The general solution is $P(t) = \frac{M}{1 + Ce^{-kt}}$ where $C = \frac{M}{P(0)} - 1$.

Armin Straub
straub@southalabama.edu

Example 46. Let $P(t)$ describe the size of a population at time t. Under the logistic model of population growth, what is $\lim_{t\to\infty} P(t)$?

Solution.

- If $k>0$, then $e^{-kt}\to 0$ and it follows from $P(t)=\frac{M}{1+Ce^{-kt}}$ that $\lim_{t\to\infty} P(t)=M$.

  In other words, the population will approach the carrying capacity in the long run.

- If $k=0$, then we simply have $P(t)=\frac{M}{1+C}$. In other words, the population remains constant.

  This is a corner case because the DE becomes $\frac{\mathrm{d}P}{\mathrm{d}t}=0$.

- If $k<0$, then $e^{-kt}\to\infty$ and it follows that $\lim_{t\to\infty} P(t)=0$.

  In other words, the population will approach extinction in the long run.

Example 47. (homework) A rising population is modeled by the equation $\frac{\mathrm{d}P}{\mathrm{d}t}=400P-2P^2$.

(a) When the population size stabilizes in the long term, how big will the population be?

(b) Under which condition will the population size shrink?

(c) What is the population size when it is growing the fastest?

(d) If $P(0)=10$, what is $P(t)$?

Solution.

(a) Once the population reaches a stable level in the long term, we have $\frac{\mathrm{d}P}{\mathrm{d}t}=0$ (no change in population size).
Hence, $0=400P-2P^2=2P(200-P)$ which implies that $P=0$ or $P=200$. Since the population is rising, it will approach 200 in the long term.

Alternatively. Our DE matches the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t}=kP\left(1-\frac{P}{M}\right)$ with $k=400$ and $M=200$.

(b) The population size will shrink if $\frac{\mathrm{d}P}{\mathrm{d}t}<0$.
The DE tells us that is the case if and only if $400P-2P^2<0$ or, equivalently, if $P>\frac{400}{2}=200$.

Comment. In the logistic model, the population shrinks if it exceeds the carrying capacity.

(c) This is asking when $\frac{\mathrm{d}P}{\mathrm{d}t}$ (the population growth) is maximal.
The DE is telling us that this growth is $f(P)=400P-2P^2$. This a parabola that opens to the bottom. From Calculus, we know that it has a global maximum when $f'(P)=0$.
$f'(P)=400-4P=0$ leads to $P=100$.
Thus, the population is growing the fastest when its size is 100.

Comment. In the logistic model, the population is growing fastest when it is half the carrying capacity.

(d) We know that the general solution of the logistic equation is $P(t)=\frac{M}{1+Ce^{-kt}}=\frac{200}{1+Ce^{-400t}}$.
Using $P(0)=10$, we find that $C=\frac{200}{10}-1=19$.
Thus $P(t)=\frac{200}{1+19e^{-400t}}$.

Example 48. A scientist is claiming that a certain population $P(t)$ follows the logistic model of population growth. How many data points do you need to begin to verify that claim?

Solution. The general solution $P(t)=\frac{M}{1+Ce^{-kt}}$ to the logistic equation has 3 parameters.

Hence, we need 3 data points just to solve for their values.

Once we have 4 or more data points, we are able to test whether $P(t)$ conforms to the logistic model.

Important comment. Complicated models tend to have many degrees of freedom, which makes it easier to fit them to real world data (even if the model is not actually particularly appropriate). We therefore need to be cognizant about how much evidence is needed to decide that a given model is appropriate for the data.