Solving simple 2nd order DEs

We have the following two useful substitutions for certain simple DEs of order 2:

- $F(y'', y', x) = 0$ (2nd order with "y missing")

  Set $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx}$. We get the first-order DE $F\left(\frac{du}{dx}, u, x\right) = 0$.

- $F(y'', y', y) = 0$ (2nd order with "x missing")

  Set $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u$. We get the first-order DE $F\left(u\frac{du}{dy}, u, y\right) = 0$.

Example 40. Solve $y'' = x - y'$.

Solution. We substitute $u = y'$, which results in the first-order DE $u' = x - u$.
This DE is linear and, using our recipe (see below for the details), we can solve it to find $u = x - 1 + Ce^{-x}$.
Since $y' = u$, we conclude that the general solution is $y = \int (x - 1 + Ce^{-x})dx = \frac{1}{2}x^2 - x - Ce^{-x} + D$.

Important comment. This is a DE of order 2. Hence, as expected, the general solution has two free parameter.

Solving the linear DE. To solve $u' = x - u$ (also see Example 30, where we had solved this DE before), we

(a) rewrite the DE as $\frac{du}{dx} + P(x)u = Q(x)$ with $P(x) = 1$ and $Q(x) = x$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)dx\right) = e^x$.

(c) Multiply the (rewritten) DE by $f(x) = e^x$ to get $\underbrace{e^x\frac{du}{dx} + e^x u}_{= \frac{d}{dx}[e^x u]} = xe^x$.

(d) Integrate both sides to get (using integration by parts): $e^x u = \int xe^x dx = xe^x - e^x + C$

Hence, the general solution of the DE for u is $u = x - 1 + Ce^{-x}$, which is what we used above.

Example 41. (homework) Solve the IVP $y'' = x - y'$, $y(0) = 1$, $y'(0) = 2$.

Solution. As in the previous example, we find that the general solution to the DE is $y(x) = \frac{1}{2}x^2 - x - Ce^{-x} + D$.
Using $y'(x) = x - 1 + Ce^{-x}$ and $y'(0) = 2$, we find that $2 = -1 + C$. Hence, $C = 3$.
Then, using $y(x) = \frac{1}{2}x^2 - x - 3e^{-x} + D$ and $y(0) = 1$, we find $1 = -3 + D$. Hence, $D = 4$.
In conclusion, the unique solution to the IVP is $y(x) = \frac{1}{2}x^2 - x - 3e^{-x} + 4$.

Example 42. (extra) Find the general solution to $y'' = 2yy'$.

Solution. We substitute $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u$.
Therefore, our DE turns into $u\frac{du}{dy} = 2yu$.
Dividing by u, we get $\frac{du}{dy} = 2y$. [Note that we lose the solution $u = 0$, which gives the singular solution $y = C$.]
Hence, $u = y^2 + C$. It remains to solve $y' = y^2 + C$. This is a separable DE.
$\frac{1}{C + y^2}dy = dx$. Let us restrict to $C = D^2 \geqslant 0$ here. (This means we will only find "half" of the solutions.)
$\int \frac{1}{D^2 + y^2}dy = \frac{1}{D^2}\int \frac{1}{1 + (y/D)^2}dy = \frac{1}{D}\arctan(y/D) = x + A$.
Solving for y, we find $y = D\tan(Dx + AD) = D\tan(Dx + B)$. $[B = AD]$

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Applications of DEs & Modeling

The exponential model of population growth

If $P(t)$ is the size of a population (eg. of bacteria) at time t, then the rate of change $\frac{\mathrm{d}P}{\mathrm{d}t}$ might, from biological considerations, be (nearly) proportional to $P(t)$.

Why? This might be more clear if we use some (random) numbers. Say, we have a population of $P=100$ and $P'=3$, meaning that the population changes by 3 individuals per unit of time. By how do we expect a population of $P=500$ to change? (Think about it for a moment!)

[Without further information, we would probably expect the population of $P=500$ to change by $5\cdot 3=15$ individuals per unit of time, so that $P'=15$ in that case. This is what it means for P' to be proportional to P. In formulas, it means that P'/P is constant or, equivalently, that $P'=kP$ for a proportionality constant k.]

Comment. "Population" might sound more specific than it is. It could also refer to rather different populations such as amounts of money (finance) or amounts of radioactive material (physics).

For instance, thinking about an amount $P(t)$ of money in a bank account at time t, we would also expect $\frac{\mathrm{d}P}{\mathrm{d}t}$ (the money per time that we gain from receiving interest) to be proportional to $P(t)$.

The corresponding **mathematical model** is described by the DE $\frac{\mathrm{d}P}{\mathrm{d}t}=kP$ where k is the constant of proportionality.

Example 43. Determine all solutions to the DE $\frac{\mathrm{d}P}{\mathrm{d}t}=kP$.

Solution. We easily guess and then verify that $P(t)=Ce^{kt}$ is a solution. (Alternatively, we can find this solution via separation of variables or because this is a linear DE. Do it both ways!)

Moreover, it follows from the existence and uniqueness theorem that there cannot be further solutions. (Alternatively, we can conclude this from our solving process (separation of variables or our approach to linear DEs only lose solutions when we divide by zero and we can consider those cases separately)).

Mathematics therefore tells us that the (only) solutions to this DE are given by $P(t)=Ce^{kt}$ where C is some constant.

Hence, populations satisfying the assumption from biology necessarily exhibit exponential growth.

Example 44. Let $P(t)$ describe the size of a population at time t. Suppose $P(0)=100$ and $P(1)=300$. Under the exponential model of population growth, find $P(t)$.

Solution. $P(t)$ solves the DE $\frac{\mathrm{d}P}{\mathrm{d}t}=kP$ and therefore is of the form $P(t)=Ce^{kt}$.

We now use the two data points to determine both C and k.

$Ce^{k\cdot 0}=C=100$ and $Ce^{k}=100e^{k}=300$. Hence $k=\ln(3)$ and $P(t)=100e^{\ln(3)t}=100\cdot 3^{t}$.

Main challenge of modeling: a model has to be detailed enough to resemble the real world, yet simple enough to allow for mathematical analysis.

Observe that the exponential model of population growth can be written as

$$\frac{P'}{P}=\text{constant}.$$

Thinking purely mathematically (generally not a good idea for modeling!), to extend the model, it might be sensible to replace constant (which we called k above) by the next simplest kind of function, namely a linear function in P. The resulting

Comment. Can you put into words why we replace constant by a function of P rather than a function of t? When would it be appropriate to add a dependence on t?

[A dependence on t would make sense if the "environment" changes over time. Without such a change, we expect that a population (say, of bacteria in our lab) behaves this week just as it would next week. The "law" behind its growth should not depend on t. The resulting differential equations are called **autonomous**.]