Example 36. (homework) Consider the DE $x\frac{dy}{dx} = y + y^2 f(x)$.

(a) Substitute $u = \frac{y}{x}$. Is the resulting DE separable or linear?

(b) Substitute $v = \frac{1}{y}$. Is the resulting DE separable or linear?

(c) Solve each of the new DEs.

Solution.

(a) Set $u = \frac{y}{x}$. Then $y = ux$ and, thus, $\frac{dy}{dx} = x\frac{du}{dx} + u$.

Using these, the DE translates into $x\left(x\frac{du}{dx} + u\right) = ux + (ux)^2 f(x)$.

This DE simplifies to $\frac{du}{dx} = u^2 f(x)$. This is a separable DE.

(b) Set $v = \frac{1}{y}$. Then $y = \frac{1}{v}$ and, thus, $\frac{dy}{dx} = -\frac{1}{v^2}\frac{dv}{dx}$.

Using these, the DE translates into $x\left(-\frac{1}{v^2}\frac{dv}{dx}\right) = \frac{1}{v} + \frac{1}{v^2} f(x)$.

This DE simplifies to $x\frac{dv}{dx} = -v - f(x)$. This is a linear DE.

(c) Let us write $F(x)$ for an antiderivative of $f(x)$.

- The DE $\frac{du}{dx} = u^2 f(x)$ from the first part is separable: $u^2 du = f(x) dx$.
  After integration, we find $-\frac{1}{u} = F(x) + C$.
  Since $u = \frac{y}{x}$, this becomes $-\frac{x}{y} = F(x) + C$.
  The general solution of the initial DE therefore is $y = -\frac{x}{F(x) + C}$.

- The DE $x\frac{dv}{dx} = -v - f(x)$ from the second part is linear. We apply our recipe:

  (a) Rewrite the DE as $\frac{dv}{dx} + P(x)v = Q(x)$ with $P(x) = 1/x$ and $Q(x) = -f(x)/x$.

  (b) The integrating factor is $\exp\left(\int P(x) dx\right) = e^{\ln x} = x$.

  Comment. We should make a mental note that we assumed that $x > 0$. In the next step, however, we see that the integrating factor works for all x.

  (c) Multiply the (rewritten) DE by the integrating factor x to get $\underbrace{x\frac{dv}{dx} + v}_{= \frac{d}{dx}[xv]} = -f(x)$.

  (d) Integrate both sides to get $xv = -F(x) + C$.

  Since $v = \frac{1}{y}$, we find $\frac{x}{y} = -F(x) + C$.
  The general solution of the initial DE therefore is $y = -\frac{x}{F(x) - C}$.

Comment. Note that our two approaches led to the same general solution (from the existence and uniqueness theorem, we can see that this must be the case). One of the formulas features $+C$ while the other features $-C$. However, that makes no difference because C is a free parameter (we could have given them different names if we preferred).

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Useful substitutions

The previous example illustrates that different substitutions can help to solve a given DE.

Choosing the right substitution is difficult in general. The following is a compilation of important cases that are easy to spot and for which the listed substitutions are guaranteed to succeed:

- $y' = F\left(\frac{y}{x}\right)$

  Set $u = \frac{y}{x}$. Then $y = ux$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u$. We get $x\frac{\mathrm{d}u}{\mathrm{d}x} + u = F(u)$. This DE is always separable.

  Caution. The DE $y' = F\left(\frac{y}{x}\right)$ is sometimes called a "homogeneous equation". However, we will soon discuss homogeneous linear differential equations, where the label homogeneous means something different (though in both cases, there is a common underlying reason).

- $y' = F(ax + by)$

  Set $u = ax + by$. Then $y = \frac{1}{b}(u - ax)$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{b}\left(\frac{\mathrm{d}u}{\mathrm{d}x} - a\right)$.

  The new DE is $\frac{1}{b}\left(\frac{\mathrm{d}u}{\mathrm{d}x} - a\right) = F(u)$ or, simplified, $\frac{\mathrm{d}u}{\mathrm{d}x} = a + bF(u)$. This DE is always separable.

- $y' = F(x)y + G(x)y^n$ (This is called a **Bernoulli equation**.)

  Set $u = y^{1-n}$. The resulting DE is always linear.

  Details. If $u = y^{1-n}$ then $y = u^{1/(1-n)}$ and, thus, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1-n}u^{n/(1-n)}\frac{\mathrm{d}u}{\mathrm{d}x}$. $\left[\frac{1}{1-n} - 1 = \frac{n}{1-n}\right]$

  The new DE is $\frac{1}{1-n}u^{n/(1-n)}\frac{\mathrm{d}u}{\mathrm{d}x} = F(x)u^{1/(1-n)} + G(x)u^{n/(1-n)}$.

  Dividing both sides by $u^{n/(1-n)}$, the DE simplifies to $\frac{1}{1-n}\frac{\mathrm{d}u}{\mathrm{d}x} = F(x)u + G(x)$ which is a linear DE.

  Comment. The original DE has the trivial solution $y = 0$. Do you see where we lost that solution?

Example 37. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = (2x - 3y)^2 + \frac{2}{3}$, $y(1) = \frac{1}{3}$.

Solution. This is of the form $y' = F(2x - 3y)$ with $F(t) = t^2 + \frac{2}{3}$.

Therefore, as suggested by the above list, we substitute $u = 2x - 3y$.

Then $y = \frac{1}{3}(2x - u)$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}\left(2 - \frac{\mathrm{d}u}{\mathrm{d}x}\right)$.

The new DE is $\frac{1}{3}\left(2 - \frac{\mathrm{d}u}{\mathrm{d}x}\right) = u^2 + \frac{2}{3}$ or, simplified, $\frac{\mathrm{d}u}{\mathrm{d}x} = -3u^2$.

This DE is separable: $u^{-2}\mathrm{d}u = -3\mathrm{d}x$. After integration, $-\frac{1}{u} = -3x + C$.

We conclude that $u = \frac{1}{3x - C}$ and, hence, $y(x) = \frac{1}{3}(2x - u) = \frac{2}{3}x - \frac{1}{3}\frac{1}{3x - C}$.

Solving $y(1) = \frac{2}{3} - \frac{1}{3}\frac{1}{3 - C} = \frac{1}{3}$ for C leads to $C = 2$.

Hence, the unique solution of the IVP is $y(x) = \frac{2}{3}x - \frac{1}{3(3x - 2)}$.

Example 38. Solve $(x - y)\frac{\mathrm{d}y}{\mathrm{d}x} = x + y$.

Solution. Divide the DE by x to get $\left(1 - \frac{y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \frac{y}{x}$. This is a DE of the form $y' = F\left(\frac{y}{x}\right)$.

We therefore substitute $u = \frac{y}{x}$. Then $y = ux$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u$.

The resulting DE is $(x - ux)\left(x\frac{\mathrm{d}u}{\mathrm{d}x} + u\right) = x + ux$, which simplifies to $x(1 - u)\frac{\mathrm{d}u}{\mathrm{d}x} = 1 + u^2$.

This DE is separable: $\frac{1 - u}{1 + u^2}\mathrm{d}u = \frac{1}{x}\mathrm{d}x$

Integrating both sides, we find $\arctan(u) - \frac{1}{2}\ln(1 + u^2) = \ln|x| + C$.

Setting $u = y/x$, we get the (general) implicit solution $\arctan(y/x) - \frac{1}{2}\ln(1 + (y/x)^2) = \ln|x| + C$.

Comment. We used $\int \frac{1}{1 + u^2}\mathrm{d}u = \arctan(u) + C$ and $\int \frac{x}{1 + x^2}\mathrm{d}x = \frac{1}{2}\ln(1 + x^2) + C$ when integrating.

See Example 34 where we reviewed these integrals.

Example 39. Solve the IVP $\frac{dy}{dx} = 2y - 3xy^5$, $y(0) = 1$.

Solution. This is an example of a Bernoulli equation (with $n = 5$). We therefore substitute $u = y^{1-n} = y^{-4}$.
Accordingly, $y = u^{-1/4}$ and, thus, $\frac{dy}{dx} = -\frac{1}{4}u^{-5/4}\frac{du}{dx}$.
The new DE is $-\frac{1}{4}u^{-5/4}\frac{du}{dx} = 2u^{-1/4} - 3xu^{-5/4}$, which simplifies to $\frac{du}{dx} = -8u + 12x$.
This is a linear first-order DE, which we solve according to our recipe:

(a) Rewrite the DE as $\frac{du}{dx} + P(x)u = Q(x)$ with $P(x) = 8$ and $Q(x) = 12x$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)dx\right) = e^{8x}$.

(c) Multiply the (rewritten) DE by $f(x) = e^{8x}$ to get

$$\underbrace{e^{8x}\frac{du}{dx} + 8e^{8x}u}_{= \frac{d}{dx}[e^{8x}u]} = 12xe^{8x}.$$

(d) Integrate both sides to get:

$$e^{8x}u = 12\int xe^{8x}dx = 12\left(\frac{1}{8}xe^{8x} - \frac{1}{8^2}e^{8x}\right) + C = \frac{3}{2}xe^{8x} - \frac{3}{16}e^{8x} + C$$

Here we used that $\int xe^{ax}dx = \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax}$. (Integration by parts!)

The general solution of the DE for u therefore is $u = \frac{3}{2}x - \frac{3}{16} + Ce^{-8x}$.
Correspondingly, the general solution of the initial DE is $y = u^{-1/4} = 1/\sqrt[4]{\frac{3}{2}x - \frac{3}{16} + Ce^{-8x}}$.
Using $y(0) = 1$, we find $1 = 1/\sqrt[4]{C - \frac{3}{16}}$ from which we obtain $C = 1 + \frac{3}{16} = \frac{19}{16}$.
The unique solution to the IVP therefore is $y = 1/\sqrt[4]{\frac{3}{2}x - \frac{3}{16} + \frac{19}{16}e^{-8x}}$.