Example 32. (review) Solve $xy' = 2y + 1$, $y(-2) = 0$.

Solution. This is a linear first-order DE.

(a) Rewrite the DE as $\frac{dy}{dx} + P(x)y = Q(x)$ with $P(x) = -\frac{2}{x}$ and $Q(x) = \frac{1}{x}$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)dx\right) = e^{-2\ln|x|} = e^{-2\ln(-x)} = (-x)^{-2} = \frac{1}{x^2}$.

Here, we used that, at least locally, $x < 0$ (because the initial condition is $x = -2 < 0$) so that $|x| = -x$.

(c) Multiply the DE (in standard form) by $f(x) = \frac{1}{x^2}$ to get

$$\underbrace{\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3}y}_{=\frac{d}{dx}\left[\frac{1}{x^2}y\right]} = \frac{1}{x^3}.$$

(d) Integrate both sides to get

$$\frac{1}{x^2}y = \int \frac{1}{x^3}dx = -\frac{1}{2x^2} + C.$$

Hence, the general solution is $y(x) = -\frac{1}{2} + Cx^2$.

Solving $y(-2) = -\frac{1}{2} + 4C = 0$ for C yields $C = \frac{1}{8}$. Thus, the (unique) solution to the IVP is $y(x) = \frac{1}{8}x^2 - \frac{1}{2}$.

Example 33. (extra) Solve $y' = 2y + 3x - 1$, $y(0) = 2$.

Solution. This is a linear first-order DE.

(a) Rewrite the DE as $\frac{dy}{dx} + P(x)y = Q(x)$ with $P(x) = -2$ and $Q(x) = 3x - 1$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)dx\right) = e^{-2x}$.

(c) Multiply the DE (in standard form) by $f(x) = e^{-2x}$ to get

$$\underbrace{e^{-2x}\frac{dy}{dx} - 2e^{-2x}y}_{=\frac{d}{dx}[e^{-2x}y]} = (3x-1)e^{-2x}.$$

(d) Integrate both sides to get

$$\begin{aligned} e^{-2x}y &= \int (3x-1)e^{-2x}dx \\ &= 3\int xe^{-2x}dx - \int e^{-2x}dx \\ &= 3\left(-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right) - \left(-\frac{1}{2}e^{-2x}\right) + C \\ &= -\frac{3}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C. \end{aligned}$$

Here, we used that $\int xe^{-2x}dx = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$ (for instance, via integration by parts with $f(x) = x$ and $g'(x) = e^{-2x}$).

Hence, the general solution is $y(x) = -\frac{3}{2}x - \frac{1}{4} + Ce^{2x}$.

Solving $y(0) = -\frac{1}{4} + C = 2$ for C yields $C = \frac{9}{4}$.

In conclusion, the (unique) solution to the IVP is $y(x) = -\frac{3}{2}x - \frac{1}{4} + \frac{9}{4}e^{2x}$.

Substitutions in DEs

Example 34. (review) Using substitution, compute $\int \frac{x}{1+x^2}\mathrm{d}x$.

Solution. We substitute $t = 1 + x^2$. In that case, $\mathrm{d}t = 2x\,\mathrm{d}x$.

$\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\int \frac{1}{t}\mathrm{d}t = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln(1+x^2) + C$

Comment. Why were we allowed to drop the absolute value in the logarithm?

Review. On the other hand, recall that $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan(x) + C$.

Example 35. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = (x+y)^2$.

First things first. Is this DE separable? Is it linear? (No to both but make sure that this is clear to you.) This means that our previous techniques are not sufficient to solve this DE.

Solution. Looking at the right-hand side, we have a feeling that the substitution $u = x + y$ might simplify things.

Then $y = u - x$ and, therefore, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} - 1$.

Using these, the DE translates into $\frac{\mathrm{d}u}{\mathrm{d}x} - 1 = u^2$. This is a separable DE: $\frac{1}{1+u^2}\mathrm{d}u = \mathrm{d}x$

After integration, we find $\arctan(u) = x + C$ and, thus, $u = \tan(x + C)$.

The solution of the original DE is $y = u - x = \tan(x + C) - x$.

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