Linear first-order DEs

A **linear differential equation** is one where the function y and its derivatives only show up linearly (i.e. there are no terms such as y^2, $1/y$, $\sin(y)$ or $y\cdot y'$).
As such, the most general linear first-order DE is of the form

$$A(x)y' + B(x)y + C(x) = 0.$$

Comment. Note that any such DE can also be rewritten in the "**standard form**" $y' + P(x)y = Q(x)$ by dividing by $A(x)$ and rearranging. We will use this form when solving linear first-order DEs.

Example 29. (extra "warmup") Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2$.

Solution. (separation of variables) $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, $-\frac{1}{y} = x^2 + C$.

Hence the general solution is $y = \frac{1}{D - x^2}$. [There also is the singular solution $y = 0$.]

Solution. (in other words) Note that $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ can be written as $\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{1}{y}\right] = \frac{\mathrm{d}}{\mathrm{d}x}[x^2]$.
From there it follows that $-\frac{1}{y} = x^2 + C$, as above.

We now use the idea of writing both sides as a derivative to also solve linear DEs that are not separable.

The multiplication by $\frac{1}{y^2}$ will be replaced by multiplication with a so-called **integrating factor**.

Example 30. Solve $y' = x - y$.

Comment. Note that we cannot use separation of variables this time.

Solution. Rewrite the DE as $y' + y = x$.
Next, multiply both sides with e^x (we will see in a little bit how to find this "integrating factor") to get

$$\underbrace{e^x y' + e^x y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[e^x y]} = x e^x.$$

The "magic" part is that we are able to realize the new left-hand side as a derivative!
We can then integrate both sides to get

$$e^x y = \int x e^x \mathrm{d}x = x e^x - e^x + C.$$

From here it follows that $y = x - 1 + Ce^{-x}$.

Comment. For the final integral, we used that $\int x e^x \mathrm{d}x = x e^x - \int e^x \mathrm{d}x = x e^x - e^x + C$ which follows, for instance, via integration by parts (with $f(x) = x$ and $g'(x) = e^x$ in the formula reviewed below).

Review. The multiplication rule $(fg)' = f'g + fg'$ implies $fg = \int f'g + \int fg'$.
The latter is equivalent to **integration by parts**:

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x$$

Comment. Sometimes, one writes $g'(x)\mathrm{d}x = \mathrm{d}g(x)$.

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In general, we can solve any **linear first-order DE** $y' + P(x)y = Q(x)$ in this way.

- We want to multiply with an **integrating factor** $f(x)$ such that the left-hand side of the DE becomes

$$f(x)y' + f(x)P(x)y = \frac{\mathrm{d}}{\mathrm{d}x}[f(x)y].$$

  Since $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)y] = f(x)y' + f'(x)y$, we need $f'(x) = f(x)P(x)$ for that.

- Check that $f(x) = \exp\left(\int P(x)\mathrm{d}x\right)$ has this property.

  Comment. This follows directly from computing the derivative of this $f(x)$ via the chain rule.

  Homework. On the other hand, note that finding f meant solving the DE $f' = P(x)\, f$. This is a separable DE. Solve it by separation of variables to arrive at the above formula for $f(x)$ yourself.

  Just to make sure. There is no difference between $\exp(x)$ and e^x. Here, we prefer the former notation for typographical reasons.

With that integrating factor, we have the following recipe for solving any linear first-order equation:

(solving linear first-order DEs)

(a) Write the DE in the **standard form** $y' + P(x)y = Q(x)$.

(b) Compute the **integrating factor** as $f(x) = \exp\left(\int P(x)\mathrm{d}x\right)$.

[We can choose any constant of integration.]

(c) Multiply the DE from part (a) by $f(x)$ to get

$$\underbrace{f(x)y' + f(x)P(x)y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[f(x)y]} = f(x)Q(x).$$

(d) Integrate both sides to get

$$f(x)y = \int f(x)Q(x)\mathrm{d}x + C.$$

Then solve for y by dividing by $f(x)$.

Comment. For better understanding, we prefer to go through the above steps. On the other hand, we can combine these steps into the following formula for the general solution of $y' + P(x)y = Q(x)$:

$$y = \frac{1}{f(x)}\left(\int f(x)Q(x)\mathrm{d}x + C\right) \quad \text{where } f(x) = e^{\int P(x)\mathrm{d}x}$$

Existence and uniqueness. Note that the solution we construct exists on any interval on which P and Q are continuous (not just on some possibly very small interval). This is better than what the existence and uniqueness theorem (Theorem 22) can guarantee. This is one of the many ways in which linear DEs have particularly nice properties compared to DEs in general.

Example 31. Solve $x^2 y' = 1 - xy + 2x$, $y(1) = 3$.

Solution. This is a linear first-order DE. We can therefore solve it according to the recipe above.

(a) Rewrite the DE as $\frac{dy}{dx} + P(x)y = Q(x)$ with $P(x) = \frac{1}{x}$ and $Q(x) = \frac{1}{x^2} + \frac{2}{x}$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)dx\right) = e^{\ln x} = x$.

Here, we could write $\ln x$ instead of $\ln|x|$ because the initial condition tells us that $x > 0$, at least locally.

Comment. We can also choose a different constant of integration but that would only complicate things.

(c) Multiply the DE (in standard form) by $f(x) = x$ to get

$$\underbrace{x\frac{dy}{dx} + y}_{= \frac{d}{dx}[xy]} = \frac{1}{x} + 2.$$

(d) Integrate both sides to get (again, we use that $x > 0$ to avoid having to use $|x|$)

$$xy = \int \left(\frac{1}{x} + 2\right)dx = \ln x + 2x + C.$$

Using $y(1) = 3$ to find C, we get $1 \cdot 3 = \ln(1) + 2 \cdot 1 + C$ which results in $C = 3 - 2 = 1$.

Hence, the (unique) solution to the IVP is $y = \frac{\ln(x) + 2x + 1}{x}$.

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