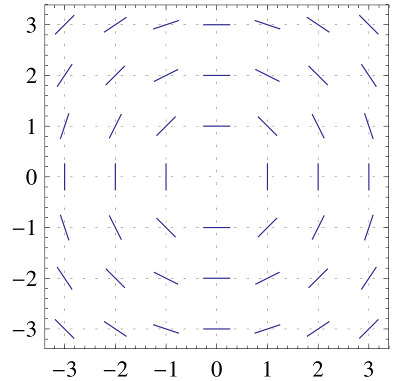
Review. Existence and uniqueness theorem (Theorem 22) for an IVP $y' = f(x, y)$, $y(a) = b$:
If $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous around (a, b) then, locally, the IVP has a unique solution.

Example 23. Consider, again, the IVP $y' = -x / y$, $y(a) = b$. Discuss existence and uniqueness of solutions (without solving).



Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = -x / y$.

We compute that $\frac{\partial}{\partial y} f(x, y) = x / y^2$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $y \neq 0$.

Hence, if $b \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

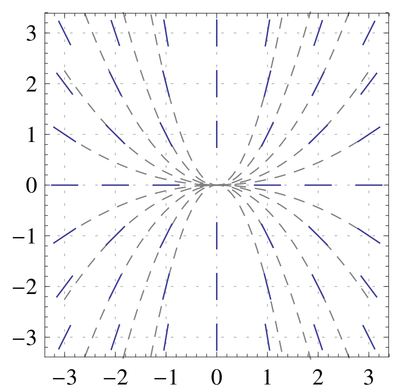
Comment. In Example 14, we found that the DE $y' = -x / y$ is solved by $y(x) = \pm\sqrt{D - x^2}$.

Assume $b > 0$ (things work similarly for $b < 0$). Then $y(x) = \sqrt{D - x^2}$ solves the IVP (we need to choose D so that $y(a) = b$) if we choose $D = a^2 + b^2$. This confirms that there exists a solution. On the other hand, uniqueness means that there can be no other solution to the IVP than this one.

What happens in the case $b = 0$?

Solution. In this case, the existence and uniqueness theorem does not guarantee anything. If $a \neq 0$, then $y(x) = \sqrt{a^2 - x^2}$ and $y(x) = -\sqrt{a^2 - x^2}$ both solve the IVP (so we certainly don't have uniqueness), however only in a weak sense: namely, both of these solutions are not valid locally around $x = a$ but only in an interval of which a is an endpoint (for instance, the IVP $y' = -x / y$, $y(2) = 0$ is solved by $y(x) = \pm\sqrt{4 - x^2}$ but both of these solutions are only valid on the interval $[-2, 2]$ which ends at 2, and neither of these solutions can be extended past 2).

Example 24. Consider, again, the IVP $xy' = 2y$, $y(a) = b$. Discuss existence and uniqueness of solutions (without solving).



Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = 2y / x$.

We compute that $\frac{\partial}{\partial y} f(x, y) = 2 / x$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $x \neq 0$.

Hence, if $a \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

What happens in the case $a = 0$?

Solution. In Example 16, we found that the DE $xy' = 2y$ is solved by $y(x) = Cx^2$.

This means that the IVP with $y(0) = 0$ has infinitely many solutions.

On the other hand, the IVP with $y(0) = b$ where $b \neq 0$ has no solutions. (This follows from the fact that there are no solutions to the DE besides $y(x) = Cx^2$. Can you see this by looking at the slope field?)

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Example 25. Consider the IVP $y' = ky^2$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x,y)$ with $f(x,y) = ky^2$. We compute that $\frac{\partial}{\partial y} f(x,y) = 2ky$.

We observe that both $f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$ are continuous for all (x,y).

Hence, for any initial conditions, the IVP locally has a unique solution by the existence and uniqueness theorem.

Example 26. Solve $y' = ky^2$.

Solution. Separate variables to get $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = k$.

Integrating $\int \frac{1}{y^2}\mathrm{d}y = \int k\mathrm{d}x$, we find $-\frac{1}{y} = kx + C$.

We solve for y to get $y = -\frac{1}{C + kx} = \frac{1}{D - kx}$ (with $D = -C$). That is the solution we verified earlier!

Comment. Note that we did not find the solution $y = 0$ (it was "lost" when we divided by y^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). However, note that we can obtain it from the general solution by letting $D \to \infty$.

Caution. We have to be careful about transforming our DE when using separation of variables: Just as the division by y^2 made us lose a solution, other transformations can add extra solutions which do not solve the original DE.

Here is a silly example (silly, because the transformation serves no purpose here) which still illustrates the point. The DE $(y-1)y' = (y-1)ky^2$ has the same solutions as $y' = ky^2$ plus the additional solution $y = 1$ (which does not solve $y' = ky^2$).

Example 27. (extra) Solve the IVP $y' = y^2$, $y(0) = 1$.

Solution. From the previous example with $k = 1$, we know that $y(x) = \frac{1}{D - x}$.

Using $y(0) = 1$, we find that $D = 1$ so that the unique solution to the IVP is $y(x) = \frac{1}{1 - x}$.

Comment. Note that we already concluded the uniqueness from the existence and uniqueness theorem.

On the other hand, note that $y(x) = \frac{1}{1 - x}$ is only valid on $(-\infty, 1)$ and that it cannot be continuously extended past $x = 1$; it is only a local solution.

Example 28. (homework) Consider the IVP $(x - y^2)y' = 3x$, $y(4) = b$. For which choices of b does the existence and uniqueness theorem guarantee a unique (local) solution?

Solution. The IVP is $y' = f(x,y)$ with $f(x,y) = 3x/(x - y^2)$. We compute that $\frac{\partial}{\partial y} f(x,y) = 6xy/(x - y^2)^2$.

We observe that both $f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$ are continuous for all (x,y) with $x - y^2 \neq 0$.

Note that $4 - b^2 \neq 0$ is equivalent to $b \neq \pm 2$.

Hence, if $b \neq \pm 2$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

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