Example 20. (homework) Consider the DE $x^2y' = 1 + xy^3$. Suppose that $y(x)$ is a solution passing through the point $(2,1)$.

(a) Determine $y'(2)$.

(b) Determine the tangent line of $y(x)$ at $(2,1)$.

(c) Determine $y''(2)$.

Comment. Note that this DE is not separable.

Solution.

(a) At the point $(2,1)$ we have $x=2$ and $y=1$. Plugging these values into the differential equation, we get $4y' = 1 + 2\cdot 1^3 = 3$ which we can solve for y' to find $y' = \frac{3}{4}$.

Since y' is short for $y'(x) = y'(2)$, we have found $y'(2) = \frac{3}{4}$.

(b) The tangent line is the line through $(2,1)$ with slope $\frac{3}{4}$ (computed in the previous part).

From this information, we can immediately write down its equation in the form $y = \frac{3}{4}(x-2) + 1$.

(c) To get our hands on $y''(2)$, we can differentiate (with respect to x) both sides of $x^2y' = 1 + xy^3$. Applying the product rule, we have $\frac{\mathrm{d}}{\mathrm{d}x}x^2y'(x) = 2xy'(x) + x^2y''(x) = 2xy' + x^2y''$ as well as $\frac{\mathrm{d}}{\mathrm{d}x}(1 + xy(x)^3) = y(x)^3 + x\cdot 3y(x)^2\cdot y'(x) = y^3 + 3xy^2y'$.

Thus $2xy' + x^2y'' = y^3 + 3xy^2y'$. To find $y''(2)$, we plug in $x=2$, $y=1$, $y' = \frac{3}{4}$.

This results in $2\cdot 2\cdot\frac{3}{4} + 4y'' = 1 + 3\cdot 2\cdot 1\cdot\frac{3}{4}$ or $3 + 4y'' = \frac{11}{2}$. It follows that $y'' = \frac{1}{4}\cdot\frac{5}{2} = \frac{5}{8}$.

Comment. Alternatively, we can rewrite the DE as $y' = \frac{1}{x^2} + \frac{1}{x}y^3$ and then differentiate. Do it!

Comment. Do you recall from Calculus what it means visually to have $y'' = \frac{5}{8}$?

[Since $y'' > 0$ it means that our function is concave up at $(2,1)$. As such, its graph will lie above the tangent line.]

Comment. Note that we could continue and likewise find $y'''(2)$ or higher derivatives at $x=2$. This is the starting point for the power series method typically discussed in Differential Equations II.

ODEs vs PDEs

Important. Note that we are working with functions $y(x)$ of a single variable. This allows us to write simply y' for $\frac{\mathrm{d}}{\mathrm{d}x}y(x)$ without risk of confusion.

Of course, we may use different variables such as $x(t)$ and $x' = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, as long as this is clear from the context.

Differential equations that involve only derivatives with respect to a single variable are known as **ordinary differential equations** (ODEs).

On the other hand, differential equations that involve derivatives with respect to several variables are referred to as **partial differential equations** (PDEs).

Example 21. The DE

$$\left(\frac{\partial}{\partial x}\right)^2 u(x,y) + \left(\frac{\partial}{\partial y}\right)^2 u(x,y) = 0,$$

often abbreviated as $u_{xx} + u_{yy} = 0$, is a partial differential equation in two variables.

This particular PDE is known as **Laplace's equation** and describes, for instance, steady-state heat distributions.

https://en.wikipedia.org/wiki/Laplace%27s_equation

This and other fundamental PDEs will be discussed in Differential Equations II.

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Existence and uniqueness of solutions

The following is a very general result that allows us to guarantee that "nice" IVPs must have a solution and that this solution is unique.

Comment. Note that any first-order DE can be written as $g(y', y, x) = 0$ where g is some function of three variables. Assuming that g is reasonable, we can solve for y' and rewrite such a DE as $y' = f(x, y)$ (for some, possibly complicated, function f).

Comment. To be precise, a solution to the IVP $y' = f(x, y)$, $y(a) = b$ is a function $y(x)$, defined on an interval I containing a, such that $y'(x) = f(x, y(x))$ for all $x \in I$ and $y(a) = b$.

Theorem 22. (existence and uniqueness) Consider the IVP $y' = f(x, y)$, $y(a) = b$.
If both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous [in a rectangle] around (a, b), then the IVP has a unique solution in some interval $x \in (a - \delta, a + \delta)$ where $\delta > 0$.

Comment. The interval around a might be very small. In other words, the δ in the theorem could be very small.

Comment. Note that the theorem makes two important assertions. First, it says that there exists a **local solution**. Second, it says that this solution is unique. These two parts of the theorem are famous results usually attributed to Peano (existence) and Picard–Lindelöf (uniqueness).

Advanced comment. The condition about $\frac{\partial}{\partial y} f(x, y)$ is a bit technical (and not optimal). If we drop this condition, we still get existence but, in general, no longer uniqueness.

Advanced comment. The interval in which the solution is unique could be smaller than the interval in which it exists. In other words, it is possible that, away from the initial condition, the solution "forks" into two or more solutions. Note that this does not contradict the theorem because it only guarantees uniqueness on a small interval.

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