

Solving DEs: Separation of variables, cont'd

In general, **separation of variables** solves $y' = g(x)h(y)$ by writing the DE as $\frac{1}{h(y)} dy = g(x)dx$.

Note that $\frac{1}{h(y)} \frac{dy}{dx} = g(x)$ is indeed equivalent to $\int \frac{1}{h(y)} dy = \int g(x) dx + C$. Why?! (Apply $\frac{d}{dx}$ to the integrals...)

Example 15. Solve the IVP $y' = -\frac{x}{y}$, $y(0) = -3$.

Comment. Instead of using what we found earlier in Example 14, we start from scratch to better illustrate the solution process (and how we can use the initial condition right away to determine the value of the constant of integration).

Solution. We separate variables to get $y dy = -x dx$.

Integrating gives $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$, and we use $y(0) = -3$ to find $\frac{1}{2}(-3)^2 = 0 + C$ so that $C = \frac{9}{2}$.

Hence, $x^2 + y^2 = 9$ is an **implicit** form of the solution.

Solving for y , we get $y = -\sqrt{9 - x^2}$ (note that we have to choose the negative sign so that $y(0) = -3$).

Comment. Note that our solution is a **local solution**, meaning that it is valid (and solves the DE) locally around $x = 0$ (from the initial condition). However, it is not a **global solution** because it doesn't make sense outside of x in the interval $[-3, 3]$.

Example 16. Consider the DE $xy' = 2y$.

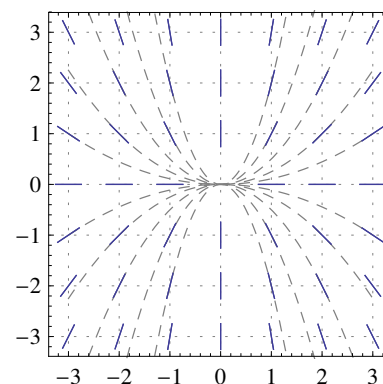
Sketch its slope field.

Challenge. Try to guess solutions $y(x)$ from the slope field.

Solution. For instance, to find the slope at the point $(3, 1)$, we plug $x = 3$, $y = 1$ into the DE to get $3y' = 2$. Hence, the slope is $y' = 2/3$.

The resulting slope field is sketched on the right.

Solution of the challenge. Trace out the solution through $(1, 1)$ (and then some other points). Their shape looks like a parabola, so that we might guess that $y(x) = Cx^2$ solves the DE. Check that this is indeed the case by plugging into the DE!



Example 17. Solve the IVP $xy' = 2y$, $y(1) = 2$.

Solution. Rewrite the DE as $\frac{1}{y} \frac{dy}{dx} = \frac{2}{x}$.

Then multiply both sides with dx and integrate both of them to get $\int \frac{1}{y} dy = \int \frac{2}{x} dx$.

Hence, $\ln|y| = 2\ln|x| + C$.

The initial condition $y(1) = 2$ tells us that, at least locally, $x > 0$ and $y > 0$. Thus $\ln(y) = 2\ln(x) + C$.

Moreover, plugging in $x = 1$ and $y = 2$, we find $C = \ln(2)$.

Solving $\ln(y) = 2\ln(x) + \ln(2)$ for y , we find $y = e^{2\ln(x) + \ln(2)} = 2x^2$.

Comment. When solving a DE or IVP, we can generally only expect to find a **local solution**, meaning that our solution might only be valid in a small interval around the initial condition (here, we can only expect $y(x)$ to be a solution for all x in an interval around 1; especially since we assumed $x > 0$ in our solution). However, we can check (do it!) that the solution $y = 2x^2$ is actually a **global solution** (meaning that it is a solution for all x , not just locally around 1).

Let's solve the same differential equation with a different choice of initial condition:

Example 18. Solve the IVP $xy' = 2y$, $y(1) = -1$.

Solution. Again, we rewrite the DE as $\frac{1}{y} \frac{dy}{dx} = \frac{2}{x}$, multiply both sides with dx , and integrate to get $\int \frac{1}{y} dy = \int \frac{2}{x} dx$.

Hence, $\ln|y| = 2\ln|x| + C$. The initial condition $y(1) = -1$ tells us that, at least locally, $x > 0$ and $y < 0$ (note that this means $|y| = -y$). Thus $\ln(-y) = 2\ln(x) + C$.

Moreover, plugging in $x = 1$ and $y = -1$, we find $C = 0$.

Solving $\ln(-y) = 2\ln(x)$ for y , we find $y = -e^{2\ln(x)} = -x^2$. We easily verify that this is indeed a global solution.

Example 19. $y' = x + y$ is a DE for which the variables cannot be separated.

No worries, very soon we will have several tools to solve this DE as well.