Example 10. (warmup) Consider the DE $y'' = y' + 6y$.

(a) Is $y(x) = e^{2x}$ a solution?

(b) Is $y(x) = e^{3x}$ a solution?

Solution.

(a) $y' = 2e^{2x}$ and $y'' = 4e^{2x}$.
Since $y' + 6y = 8e^{2x}$ is different from $y'' = 4e^{2x}$, we conclude that $y(x) = e^{2x}$ is not a solution.

(b) $y' = 3e^{3x}$ and $y'' = 9e^{3x}$.
Since $y' + 6y = 9e^{3x}$ is equal to $y'' = 9e^{3x}$, we conclude that $y(x) = e^{3x}$ is a solution of the DE.

Example 11. (cont'd) Consider the DE $y'' = y' + 6y$. For which r is e^{rx} a solution?

Solution. If $y(x) = e^{rx}$, then $y'(x) = re^{rx}$ and $y''(x) = r^2 e^{rx}$.
Plugging $y(x) = e^{rx}$ into the DE, we get $r^2 e^{rx} = re^{rx} + 6e^{rx}$ which simplifies to $r^2 = r + 6$.
This has the two solutions $r = -2$, $r = 3$. Hence e^{-2x} and e^{3x} are solutions of the DE.

In fact, we check that $Ae^{-2x} + Be^{3x}$ is a **two-parameter family** of solutions to the DE.

Important comment. It is no coincidence that the order of the DE is 2, whereas the previous example has order 1. In general, we expect a DE of order r to have a solution with r parameters.

Example 12. (extra)

Comment. In this example, we use $x(t)$ instead of $y(x)$ for the function described by the differential equation. In general, of course, any choice of variable names is possible. If we write something like x' or y' it needs to be clear from the context with respect to which variable that derivative is meant (such as $x' = \frac{\mathrm{d}}{\mathrm{d}t} x(t)$).

(a) Verify that $x(t) = \frac{1}{c - kt}$ is a one-parameter family of solutions to the DE $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$.

(b) Solve the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$, $x(0) = 2$.

(c) Solve the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$, $x(0) = 0$.

Solution.

(a) We compute that $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{(c - kt)^2} \cdot (-k) = \frac{k}{(c - kt)^2}$.

On the other hand, $kx^2 = k\left(\frac{1}{c - kt}\right)^2 = \frac{k}{(c - kt)^2}$ as well. Thus, indeed, $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$.

(b) We start with $x(t) = \frac{1}{c - kt}$ (which we know solves the DE for any value of c) and seek to choose c so that $x(0) = 2$.
Since $x(0) = \left[\frac{1}{c - kt}\right]_{t=0} = \frac{1}{c} \stackrel{!}{=} 2$, we find $c = \frac{1}{2}$.
Hence, the IVP has the (unique) solution $x(t) = \frac{1}{1/2 - kt}$.

(c) Proceeding as in the previous part, we now arrive at the impossible equation $\frac{1}{c} \stackrel{!}{=} 0$.

However, this suggests that we should consider taking $c \to \infty$ in $x(t) = \frac{1}{c - kt}$, which results in $x(t) = 0$.

Indeed, it is easy to verify (make sure you know what this entails!) that $x(t) = 0$ solves the IVP.

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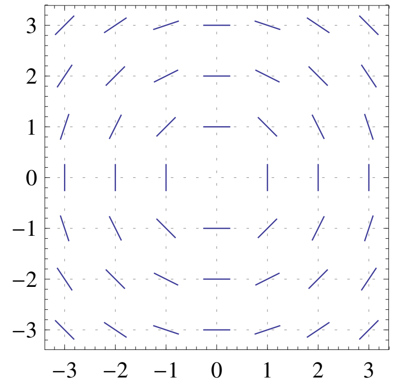
Slope fields, or sketching solutions to DEs

Example 13. Consider the DE $y' = -x/y$.

Let's pick a point, say, $(1, 2)$. If a solution $y(x)$ is passing through that point, then its slope has to be $y' = -1/2$. We therefore draw a small line through the point $(1, 2)$ with slope $-1/2$. Continuing in this fashion for several other points, we obtain the **slope field** on the right.



With just a little bit of imagination, we can now anticipate the solutions to look like (half)circles around the origin. Let us check whether $y(x) = \sqrt{r^2 - x^2}$ might indeed be a solution!

$y'(x) = \frac{1}{2}\frac{-2x}{\sqrt{r^2 - x^2}} = -x/y(x)$. So, yes, we actually found solutions!

Solving DEs: Separation of variables

Example 14. Solve the DE $y' = -\frac{x}{y}$.

Solution. Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

Separate the variables to get $y\,\mathrm{d}y = -x\,\mathrm{d}x$ (in particular, we are multiplying both sides by $\mathrm{d}x$).

Integrating both sides, we get $\int y\,\mathrm{d}y = \int -x\,\mathrm{d}x$.

Computing both integrals results in $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$ (we combine the two constants of integration into one).

Hence $x^2 + y^2 = D$ (with $D = 2C$).

This is an **implicit form** of the solutions to the DE. We can make it explicit by solving for y. Doing so, we find $y(x) = \pm\sqrt{D - x^2}$ (choosing $+$ gives us the upper half of a circle, while the negative sign gives us the lower half).

Comment. The step above where we break $\frac{\mathrm{d}y}{\mathrm{d}x}$ apart and then integrate may sound sketchy!

However, keep in mind that, after we find a solution $y(x)$, even if by sketchy means, we can (and should!) verify that $y(x)$ is indeed a solution by plugging into the DE. We actually already did that in the previous example!

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