Example 5. (cont'd) Determine several (random) DEs that $y(x)=\sin(3x)$ solves.

Solution.

(a) We compute $y'(x)=3\cos(3x)$.
Accordingly, $y(x)=\sin(3x)$ solves the DE $y'=3\cos(3x)$.
Comment. Note that there are further solutions to this DE: the **general solution** is $\int 3\cos(3x)\mathrm{d}x=\sin(3x)+C$ where C is any constant. We say that $y(x)=\sin(3x)+C$ is a **one-parameter family** of solutions to the DE. C is called a **degree of freedom**.

(b) As last time, we note that $\cos(3x)=3\sqrt{1-(\sin(3x))^2}=3\sqrt{1-y(x)^2}$ (for x close to 0).

Hence, $y(x)=\sin(3x)$ solves the differential equation $y'=3\sqrt{1-y^2}$ (for x close to 0).

(c) We compute $y''(x)=-9\sin(3x)$.
Accordingly, $y(x)=\sin(3x)$ solves the DE $y''=-9\sin(3x)$.
Comment. Once more this DE is easy (because it only involves y'' but not y or y'). Hence, we can find the general solution by simply taking two antiderivatives:

$$y(x)=\iint -9\sin(3x)\mathrm{d}x\,\mathrm{d}x=\int(3\cos(3x)+C)\mathrm{d}x=\sin(3x)+Cx+D.$$

Here it is important that we give the second constant of integration a name different from the first. That way, we see that the general solution has 2 degrees of freedom. This matches the fact that the order of the DE is 2.
Important comment. This is no coincidence. In general, we expect a DE of order r to have a general solution with r parameters.

(d) $y(x)=\sin(3x)$ also solves the DE $y''=-9y$.
Comment. This is again a DE of order 2. Therefore the general solution should have 2 degrees of freedom. Later we will learn to solve such DEs. For now, we can verify that $y(x)=A\sin(3x)+B\cos(3x)$ is a solution for any constants A and B.
Homework. Check that $y(x)=\sin(3x)+C$ does not solve the DE $y''=-9y$.

Example 6. Consider the DE $e^y y'=1$.

(a) Is $y(x)=\ln(x)$ a solution to the DE?

(b) Is $y(x)=\ln(x)+C$ a solution to the DE?

(c) Is $y(x)=\ln(x+C)$ a solution to the DE?

Solution.

(a) Since $y'(x)=\frac{1}{x}$ and $e^{y(x)}=e^{\ln(x)}=x$, we have $e^y y'=x\cdot\frac{1}{x}\overset{\checkmark}{=}1$.
Hence, $y(x)=\ln(x)$ is a solution to the given DE.

(b) Since $y'(x)=\frac{1}{x}$ and $e^{y(x)}=e^{\ln(x)+C}=xe^C$, we have $e^y y'=xe^C\cdot\frac{1}{x}=e^C$. Thus the DE is satisfied only if $e^C=1$ which only happens if $C=0$ (which is the case in the first part).
Hence, $y(x)=\ln(x)+C$ is not a solution to the given DE except if $C=0$.

(c) Since $y'(x)=\frac{1}{x+C}$ and $e^{y(x)}=e^{\ln(x+C)}=x+C$, we have $e^y y'=(x+C)\cdot\frac{1}{x+C}\overset{\checkmark}{=}1$.
Hence, $y(x)=\ln(x+C)$ is indeed a one-parameter family of solutions to the given DE.

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Example 7. Solve the DE $y' = x^2 + x$.

Solution. Note that the DE simply asks for a function $y(x)$ with a specific derivative (in particular, the right-hand side does not involve $y(x)$). In other words, the desired $y(x)$ is an **antiderivative** of $x^2 + x$. We know from Calculus II that we can find antiderivatives by integrating:

$$y(x) = \int (x^2 + x)\mathrm{d}x = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

Moreover, we know from Calculus II that there are no other solutions. In other words, we found the **general solution** to the DE.

To single out a **particular solution**, we need to specify additional conditions (typically one condition per parameter in the general solution). For instance, it is common to impose **initial conditions** such as $y(1) = 2$. A DE together with an initial condition is called an **initial value problem** (IVP).

Example 8. Solve the IVP $y' = x^2 + x$ with $y(1) = 2$.

Solution. From the previous example, we know that $y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$.

Since $y(1) = \frac{1}{3} + \frac{1}{2} + C = \frac{5}{6} + C \overset{!}{=} 2$, we find $C = 2 - \frac{5}{6} = \frac{7}{6}$.

Hence, $y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{7}{6}$ is the (unique) solution of the IVP.

Example 9. (homework) Solve the DE $y'' = x^2 + x$.

Solution. We now take two antiderivatives of $x^2 + x$ to get

$$y(x) = \iint (x^2 + x)\mathrm{d}x\mathrm{d}x = \int \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + C\right)\mathrm{d}x = \frac{1}{12}x^4 + \frac{1}{6}x^3 + Cx + D,$$

where it is important that we give the second constant of integration a name different from the first.

Important comment. Again, this is the general solution to the DE. The DE is of order 2 and, as expected, the general solution has 2 parameters.