Notes for Lecture 1

Wed, 8/23/2023

Review: Computing derivatives

Given a function $y(x)$, we learned in Calculus I that its **derivative**

$$y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

(where $\Delta y = y(x + \Delta x) - y(x)$) has the following two important characterizations:

- $y'(x)$ is the **slope of the tangent line** of the graph of $y(x)$ at x, and
- $y'(x)$ is the **rate of change** of $y(x)$ at x.

Comment. Derivatives were introduced in the late 1600s by Newton and Leibniz who later each claimed priority in laying the foundations for calculus. Certainly both of them contributed mightily to those foundations.

Moreover, we learned simple rules to compute the derivative of functions:

- **(sum rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)] = f'(x) + g'(x)$
- **(product rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
- **(chain rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x)$

  Comment. If we write $t = g(x)$ and $y = f(t)$, then the chain rule takes the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x}$.
  In other words, the chain rule expresses the fact that we can treat $\frac{\mathrm{d}y}{\mathrm{d}x}$ (which initially is just a notation for $y'(x)$) as an honest fraction.

- **(basic functions)** $\frac{\mathrm{d}}{\mathrm{d}x} x^r = r x^{r-1}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$, $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} \sin(x) = \cos(x)$, $\frac{\mathrm{d}}{\mathrm{d}x} \cos(x) = -\sin(x)$

These rules are enough to compute the derivative of any function that we can build from the basic functions using algebraic operations and composition. On the other hand, as you probably recall from Calculus II, reversing the operation of differentiation (i.e. computing antiderivatives) is much more difficult.

In particular, there exist simple functions (such as e^{x^2}) whose antiderivative cannot be expressed in terms of the basic functions above.

Example 1. Derive the **quotient rule** from the rules above.

Solution. We write $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$ and apply the product rule to get

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x) \frac{1}{g(x)} + f(x) \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{g(x)}.$$

By the chain rule combined with $\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{x} = -\frac{1}{x^2}$, we have $\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{g(x)} = -\frac{1}{g(x)^2} g'(x)$. Using this in the previous formula,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x) \frac{1}{g(x)} - f(x) \frac{1}{g(x)^2} g'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2}.$$

Putting the final two fractions on a common denominator, we obtain the familiar quotient rule

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Armin Straub
straub@southalabama.edu

Example 2. Compute the following derivatives:

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(5x^3+7x^2+2)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\sin(5x^3+7x^2+2)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}(x^3+2x)\sin(5x^3+7x^2+2)$

Solution.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(5x^3+7x^2+2)=15x^2+14x$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\sin(5x^3+7x^2+2)=(15x^2+14x)\cos(5x^3+7x^2+2)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}(x^3+2x)\sin(5x^3+7x^2+2)$
$=(3x^2+2)\sin(5x^3+7x^2+2)+(x^3+2x)(15x^2+14x)\cos(5x^3+7x^2+2)$

First examples of differential equations

Example 3. If $y(x)=e^{x^2}$ then $y'(x)=2xe^{x^2}=2xy(x)$ or, for short, $y'=2xy$.

Accordingly, we say that $y(x)=e^{x^2}$ is a **solution** to the **differential equation** (DE) $y'=2xy$.

Comment. Note that $y(x)=e^{x^2}$ also is a solution to the differential equation $y'=2xe^{x^2}$. Because this DE only involves y' but not y, we can solve it by computing an antiderivative of $2xe^{x^2}$.

Example 4.

(a) By computing its derivative, determine a DE solved by $y(x)=\sin(3x)$.

(b) By computing its second derivative, determine another DE solved by $y(x)=\sin(3x)$.

Solution.

(a) $y'(x)=3\cos(3x)=3\sqrt{1-(\sin(3x))^2}=3\sqrt{1-y(x)^2}$ (for x close to 0).

[Here we used that $\cos(x)^2+\sin(x)^2=1$, which implies that $\cos(x)=\sqrt{1-\sin(x)^2}$.]

Hence, $y(x)=\sin(3x)$ solves the differential equation $y'=3\sqrt{1-y^2}$.

Comment. In the above, we restrict x to $\left(-\frac{\pi}{6},\frac{\pi}{6}\right)$ so that $\cos(3x)>0$. Less precisely, we can say that x is close to 0. (It is a common feature of DEs that we work with values of x close to a certain initial value.)

Comment. Another possible DE would simply be $y'=3\cos(3x)$. However, that is not an "interesting" choice. In particular, this DE could be simply solved by computing an antiderivative. More next time!

(b) $y''(x)=-9\sin(3x)=-9y(x)$.

Thus, $y(x)=\sin(3x)$ also solves the differential equation $y''=-9y$.

Comment. Proceeding the same way, we can check that $y(x)=\cos(3x)$ also solves the DE $y''=-9y$. In fact, so does any linear combination $y(x)=A\cos(x)+B\sin(x)$ (where A and B are constants).

If the highest derivative appearing in a DE is an rth derivative, we say that the DE has **order** r.

For instance. The DE $y'=3\sqrt{1-y^2}$ has order 1 (such DEs are also called first order DEs).

On the other hand, the DE $y''=-9y$ has order 2 (such DEs are also called second order DEs).

Armin Straub
straub@southalabama.edu