Midterm #2

MATH 238 — Differential Equations I
Thursday, Nov 10

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 42 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (7 points) Determine the general solution $y(x)$ to the differential equation $y^{(4)} + 4y''' + 5y'' = 2$. Express the solution using real numbers only.

Solution. Since $D^4 + 4D^3 + 5D^2 = D^2(D^2 + 4D + 5)$, the characteristic equation for the corresponding homogeneous DE has roots $0, 0, -2 \pm i$ ("old" roots). The right-hand side solves a DE whose characteristic equation has root 0 ("new" root). Hence, by the method of undetermined coefficients, there must be a particular solution of the form $y_p = Ax^2$.

Plugging into the DE, we get $y_p^{(4)} + 4y_p''' + 5y_p'' = 10A \overset{!}{=} 2$. Thus $A = \frac{1}{5}$ so that $y_p = \frac{1}{5}x^2$.

Hence, the general solution is $y(x) = \frac{1}{5}x^2 + C_1 + C_2 x + C_3 e^{-2x}\cos(x) + C_4 e^{-2x}\sin(x)$.

Problem 2. (3 points) Consider a homogeneous linear differential equation with constant real coefficients which has order 6. Suppose $y(x) = 3x^2 e^{2x}\cos(x)$ is a solution. Write down the general solution.

Solution. The characteristic roots must include $2 \pm i, 2 \pm i, 2 \pm i$. Since these are 6 roots and the DE has order 6, there cannot be any additional roots.

Hence, the general solution is $(C_1 + C_2 x + C_3 x^2)e^{2x}\cos(x) + (C_4 + C_5 x + C_6 x^2)e^{2x}\sin(x)$.

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Problem 3. (10 points) Determine the general solution of the system $\begin{array}{rcl} y_1' & = & 3y_1 + y_2 \\ y_2' & = & y_1 + 3y_2 + 6e^x \end{array}$.

Solution. Using $y_2 = y_1' - 3y_1$ (from the first equation) in the second equation, we get $y_1'' - 3y_1' = y_1 + 3(y_1' - 3y_1) + 6e^x$.

Simplified, this is $y_1'' - 6y_1' + 8y_1 = 6e^x$. This is an inhomogeneous linear DE with constant coefficients. Since the "old" roots are $2, 4$, while the "new" root is 1, there must a particular solution of the form $y_1 = Ae^x$. For this y_1, $y_1'' - 6y_1' + 8y_1 = (1 - 6 + 8)Ae^x = 3Ae^x \overset{!}{=} 6e^x$. Hence, $A = 2$ and the particular solution is $y_1 = 2e^x$. The corresponding general solution is $y_1 = 2e^x + C_1e^{2x} + C_2e^{4x}$.

Correspondingly, $y_2 = y_1' - 3y_1 = 2e^x + 2C_1e^{2x} + 4C_2e^{4x} - 3(2e^x + C_1e^{2x} + C_2e^{4x}) = -4e^x - C_1e^{2x} + C_2e^{4x}$.

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Problem 4. (3 points) The position $y(t)$ of a certain mass on a spring is described by $2y'' + dy' + 3y = 0$. For which value of d is the system critically damped?

Solution. The characteristic equation has roots $\frac{1}{2}\left(-d \pm \sqrt{d^2 - 24}\right)$. The system is underdamped if the solutions involve oscillations, which happens if and only if $d^2 - 24$ (the discriminant) is negative. It is critically damped for $d = \sqrt{24}$.

Problem 5. (3 points) Write the (third-order) differential equation $y''' + 4y'' - 2y' - 3y = e^x$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, the DE translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = 3y_1 + 2y_2 - 4y_3 + e^x \end{cases}$.

In matrix form, with $\boldsymbol{y} = (y_1, y_2, y_3)$, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 2 & -4 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 0 \\ 0 \\ e^x \end{bmatrix}$.

Problem 6. (6 points) Let L be a linear differential operator of order 4 with constant real coefficients. Suppose that $2 - 3i$ is a repeated characteristic root of L.

(a) What is the general solution to $Ly = 0$?

(b) Write down the simplest form of a particular solution y_p of the DE $Ly = -4x^2 e^{3x}$ with undetermined coefficients.

(c) Write down the simplest form of a particular solution y_p of the DE $Ly = -4e^{2x}\sin(3x)$ with undetermined coefficients.

Solution. Since L is real, if $2 - 3i$ is a repeated characteristic root of L, then $2 + 3i$ must be a repeated characteristic root of L as well. Hence, the 4 characteric roots ("old" roots) must be $2 \pm 3i, 2 \pm 3i$.

(a) The general solution is $(C_1 + C_2 x)e^{2x}\cos(3x) + (C_3 + C_4 x)e^{2x}\sin(3x)$.

(b) The "new" roots are $3, 3, 3$ (no overlap with "old" roots).

Hence, there must a particular solution of the form $(C_1 + C_2 x + C_3 x^2)e^{3x}$.

(c) The "new" roots are $2 \pm 3i$ (overlap with "old" roots).

Hence, there must a particular solution of the form $C_1 x^2 e^{2x}\cos(3x) + C_2 x^2 e^{2x}\sin(3x)$.

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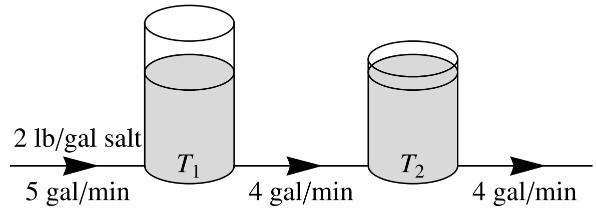
Problem 7. (6 points) The mixtures in two tanks T_1, T_2 are kept uniform by stirring. The tanks are connected as indicated in the sketch below. That is, brine containing 2 lb of salt per gallon enters T_1 at 5 gal/min, and the solution is pumped at a rate of 4 gal/min into T_2. Finally, solution is leaving T_2 at 4 gal/min.

Initially, T_1 and T_2 contain 40gal of pure water each.

Denote by $y_i(t)$ the amount (in pounds) of salt in tank T_i at time t (in minutes). Derive a system of linear differential equations for the y_i, including initial conditions.

(Do *not* attempt to solve the system.)



Solution. Note that at time t, T_1 contains $40+t$ gal of solution while T_2 contains a constant amount of 40 gal.

In the time interval $[t, t+\Delta t]$, we have:

$$\begin{aligned}
\Delta y_1 &\approx 5\cdot 2\cdot \Delta t - 4\cdot \frac{y_1}{40+t}\cdot \Delta t & \quad\Longrightarrow\quad & y_1' = 10 - \frac{4}{40+t}y_1\\
\Delta y_2 &\approx 4\cdot \frac{y_1}{40+t}\cdot \Delta t - 4\cdot \frac{y_2}{40}\cdot \Delta t & \quad\Longrightarrow\quad & y_2' = \frac{4}{40+t}y_1 - \frac{1}{10}y_2
\end{aligned}$$

We also have the initial conditions $y_1(0)=0,\ y_2(0)=0$.

Optional: in matrix form, writing $\boldsymbol{y}=(y_1, y_2)$, this takes the form

$$\boldsymbol{y}' = \begin{bmatrix} -\frac{4}{40+t} & 0 \\ \frac{4}{40+t} & -\frac{1}{10} \end{bmatrix}\boldsymbol{y} + \begin{bmatrix} 10 \\ 0 \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Problem 8. (4 points) Assume that the angle $\theta(t)$ of a swinging pendulum is described by $9\theta'' + \theta = 0$. Suppose $\theta(0) = \frac{1}{10}$, $\theta'(0) = -\frac{1}{3}$. What is the period and the amplitude of the resulting oscillations?

Solution. The characteristic equation has roots $\pm\frac{1}{3}i$. Hence, the general solution to the DE is $\theta(t) = A\cos\left(\frac{1}{3}t\right) + B\sin\left(\frac{1}{3}t\right)$.

We use the initial conditions to determine A and B: $\theta(0) = A = \frac{1}{10}$. $\theta'(0) = \frac{1}{3}B \overset{!}{=} -\frac{1}{3}$.

Hence, the unique solution to the IVP is $\theta(t) = \frac{1}{10}\cos\left(\frac{1}{3}t\right) - \sin\left(\frac{1}{3}t\right)$.

In particular, the period is 6π and the amplitude is $\sqrt{A^2+B^2} = \sqrt{\frac{1}{100}+1} = \sqrt{\frac{101}{100}} = \frac{1}{10}\sqrt{101}$.

(extra scratch paper)