Midterm #1

MATH 238 — Differential Equations I
Thursday, Sept 29

Please print your name:

No notes, calculators or tools of any kind are permitted. There are 39 points in total. You need to show work to receive full credit.

Good luck!

Problem 1. (9 points) A tank contains 10gal of pure water. It is filled with brine (containing 3lb/gal salt) at a rate of 2gal/min. At the same time, well-mixed solution flows out at a rate of 1gal/min. How much salt is in the tank after t minutes?

Solution. Let $x(t)$ denote the amount of salt (in lb) in the tank after time t (in min).

At time t, the concentration of salt (in lb/gal) in the tank is $\frac{x(t)}{V(t)}$ where $V(t) = 10 + (2-1)t = 10 + t$ is the volume (in gal) in the tank.

In the time interval $[t, t + \Delta t]$: $\Delta x \approx 2 \cdot 3 \cdot \Delta t - 1 \cdot \frac{x(t)}{V(t)} \cdot \Delta t$.

Hence, $x(t)$ solves the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = 6 - \frac{1}{10+t} x$ with $x(0) = 0$. Since this is a linear DE, we can solve it as follows:

- We write it in the form $\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{1}{10+t} x = 6$.
- The integrating factor is $f(t) = \exp\left(\int \frac{1}{10+t} \mathrm{d}t\right) = \exp(\ln(10+t)) = 10 + t$.
- Multiply the (rewritten) DE by $f(t) = 10 + t$ to get $\underbrace{(10+t)\frac{\mathrm{d}x}{\mathrm{d}t} + x}_{= \frac{\mathrm{d}}{\mathrm{d}t}[(10+t)x]} = 60 + 6t$.
- Integrate both sides to get $(10+t)x = \int (60 + 6t)\mathrm{d}t = 60t + 3t^2 + C$.

Hence the general solution to the DE is $x(t) = \frac{60t + 3t^2 + C}{10+t}$. Using $x(0) = 0$, we find $\frac{C}{10} = 0$ from which we conclude that $C = 0$.

After t minutes, the tank therefore contains $x(t) = \frac{60t + 3t^2}{10+t}$ pounds of salt.

(Depending on preference, we can also write $\frac{60t + 3t^2}{10+t} = 3(10+t) - \frac{300}{10+t}$.)

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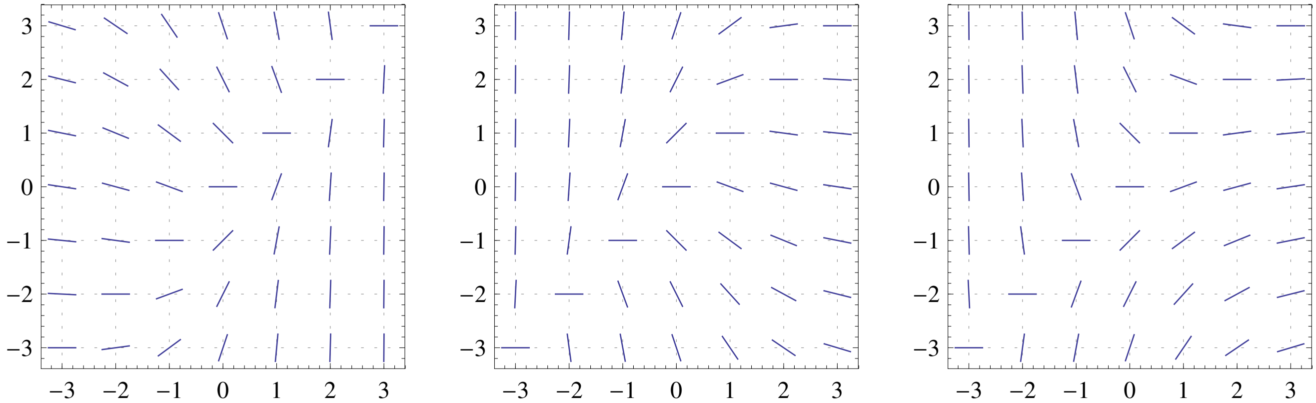
Problem 2. (2 points) Circle the slope field below which belongs to the differential equation $e^x y' = x - y$.



Solution. A good point to carefully consider is $(1, 3)$. By the DE, a solution passing through that point has slope y' satisfying $e^1 y' = 1 - 3$. Equivalently, $y' = -2/e$. The only plot compatible with that is the third one.

Of course, we can arrive at the same conclusion based on other points or, even better, based on several points.

Problem 3. (3 points) Consider the initial value problem

$$(x+2)y' = \sin(x+3y), \qquad y(a) = b.$$

For which values of a and b can we guarantee existence and uniqueness of a (local) solution?

Solution. Let us write $y' = f(x, y)$ with $f(x, y) = \frac{\sin(x+3y)}{x+2}$. Then $\frac{\partial}{\partial y} f(x, y) = \frac{3\cos(x+3y)}{x+2}$.

Both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $x \neq -2$.

Hence, if $a \neq -2$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Problem 4. (4 points) Find the general solution to the differential equation $y'' + y' = 2y$.

Solution. We look for solutions of the form e^{rx}.

Plugging e^{rx} into the DE, we get $r^2 e^{rx} + r e^{rx} = 2e^{rx}$ which simplifies to $r^2 + r - 2 = 0$.

$r^2 + r - 2 = 0$ has the two solutions $r = \frac{-1 \pm \sqrt{1 - 4 \cdot (-2)}}{2} = \frac{-1 \pm 3}{2} = -2, 1$.

This means we found the two solutions $y_1 = e^{-2x}$, $y_2 = e^x$.

The general solution to the DE is $Ae^{-2x} + Be^x$.

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Problem 5. (7 points) Determine the general solution to $x y' = y + 4x^2 e^{2x}$.

Solution. This is a linear DE. To solve it, we first bring it in the form $y' - \frac{1}{x} y = 4x e^{2x}$.

The integrating factor is $e^{\int -\frac{1}{x} \mathrm{d}x} = e^{-\ln x} = \frac{1}{x}$.

We multiply the (rewritten) DE by $\frac{1}{x}$ to get $\underbrace{\frac{1}{x} y' - \frac{1}{x^2} y}_{= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x} y\right]} = 4e^{2x}$.

We then integrate both sides to get $\frac{y}{x} = 2e^{2x} + C$.

Hence a general solution is $y = 2x e^{2x} + Cx$.

Problem 6. (2 points) A rising population is modeled by the equation $\frac{\mathrm{d}P}{\mathrm{d}t} = 1000P - 10P^2$. When the population size stabilizes in the long term, how big will the population be? Do not solve the DE!

Solution. Once the population reaches a stable level in the long term, we have $\frac{\mathrm{d}P}{\mathrm{d}t} = 0$ (no change in population size).

Hence, $0 = 1000P - 10P^2 = 10P(100 - P)$ which implies that $P = 0$ or $P = 100$. Since the population is rising, it will approach 100 in the long term.

Problem 7. (6 points) The time rate of change of a rabbit population P is proportional to the square root of P. At time $t=0$, the population numbers 25 rabbits and is increasing at the rate of 5 rabbits per month. How many rabbits will there be after two months?

Solution. $P'(t)=k\sqrt{P}$ and $P(0)=25$, $P'(0)=5$. The problem asks for $P(2)$.

At $t=0$, we have $5=P'(0)=k\sqrt{P(0)}=5k$. Hence $k=1$.

The DE $\frac{\mathrm{d}P}{\mathrm{d}t}=\sqrt{P}$ is separable: $P^{-1/2}\mathrm{d}P=\mathrm{d}t$.

Integrating both sides, we find $2\sqrt{P}=t+C$. From $P(0)=25$, we conclude that $C=10$.

Thus, $P(t)=\left(\frac{t}{2}+5\right)^2$. In particular, there will be $P(2)=6^2=36$ rabbits after two months.

Problem 8. (6 points) Solve the initial value problem $x\frac{\mathrm{d}y}{\mathrm{d}x}=y+xe^{-y/x}$, $y(1)=0$.

Solution. This DE is neither separable nor linear. Hence, we look for a suitable substitution.

Since the right-hand side features a y, we divide both sides by x. We get $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{x}+e^{-y/x}$ which is homogeneous.

We therefore substitute $u=\frac{y}{x}$. Then $y=ux$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=x\frac{\mathrm{d}u}{\mathrm{d}x}+u$.

The resulting DE is $x\frac{\mathrm{d}u}{\mathrm{d}x}+u=u+e^{-u}$, which simplifies to $x\frac{\mathrm{d}u}{\mathrm{d}x}=e^{-u}$.

This DE is separable: $e^u\,\mathrm{d}u=\frac{1}{x}\,\mathrm{d}x$. Integrating both sides, we find $e^u=\ln|x|+C$.

Accordingly, for the initial DE, $e^{y/x}=\ln(x)+C$. (Note that $x>0$, at least locally, due to the initial condition.)

Using $y(1)=0$ we find $e^{-0/1}=\ln(1)+C$ so that $C=1$.

Thus $e^{y/x}=\ln(x)+1$ and, therefore, $y=x\ln(\ln(x)+1)$.

(extra scratch paper)