Midterm #1 – Practice

MATH 238 — Differential Equations I
Midterm: Thursday, Sept 29

Please print your name:

Reminder. No notes, calculators or tools of any kind will be permitted on the midterm exam.

Problem 1.

(a) Find the general solution to $y'' + y' = 12y$.

(b) Find the general solution to $y''' + y'' = 12y'$.

Solution.

(a) We look for solutions of the form e^{rx}.

Plugging e^{rx} into the DE, we get $r^2 e^{rx} + r e^{rx} = 12 e^{rx}$ which simplifies to $r^2 + r - 12 = 0$.

This quadratic equation has the solutions $r = \frac{-1 \pm \sqrt{1 - 4 \cdot (-12)}}{2} = \frac{-1 \pm 7}{2} = -4, 3$.

This means we found the two solutions $y_1 = e^{-4x}$, $y_2 = e^{3x}$.

The general solution to the DE is $A e^{-4x} + B e^{3x}$.

(b) We look for solutions of the form e^{rx}.

Plugging e^{rx} into the DE, we get $r^3 e^{rx} + r^2 e^{rx} = 12 r e^{rx}$ which simplifies to $r^3 + r^2 - 12r = r(r^2 + r - 12) = 0$.

As before, $r^2 + r - 12 = 0$ has the solutions $r = \frac{-1 \pm \sqrt{1 - 4 \cdot (-12)}}{2} = \frac{-1 \pm 7}{2} = -4, 3$.

Overall, $r(r^2 + r - 12) = 0$ has the three solutions $-4, 3, 0$.

This means we found the three solutions $y_1 = e^{-4x}$, $y_2 = e^{3x}$, $y_3 = e^{0x} = 1$.

The general solution to the DE is $A e^{-4x} + B e^{3x} + C$.

Alternatively. We can substitute $u = y'$, in which case the new DE is $u'' + u' = 12u$. From the first part, we know that $u = A e^{-4x} + B e^{3x}$.

Hence, the general solution of the initial DE is $y = \int u \, \mathrm{d}x = -\frac{1}{4} A e^{-4x} + \frac{1}{3} B e^{3x} + C$.

Note that we can set $C_1 = -\frac{1}{4} A$, $C_2 = \frac{1}{3} B$, $C_3 = C$ to write this as $C_1 e^{-4x} + C_2 e^{3x} + C_3$ as above.

Problem 2. Consider the initial value problem

$$(xy + 2x) y' = \cos(x), \qquad y(a) = b.$$

For which values of a and b can we guarantee existence and uniqueness of a (local) solution?

Solution. Let us write $y' = f(x, y)$ with $f(x, y) = \frac{\cos(x)}{xy + 2x} = \frac{\cos(x)}{x(y + 2)}$. Then $\frac{\partial}{\partial y} f(x, y) = -\frac{\cos(x)}{x(y + 2)^2}$.

Both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $x \neq 0$ and $y \neq -2$.

Hence, if $a \neq 0$ and $b \neq -2$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

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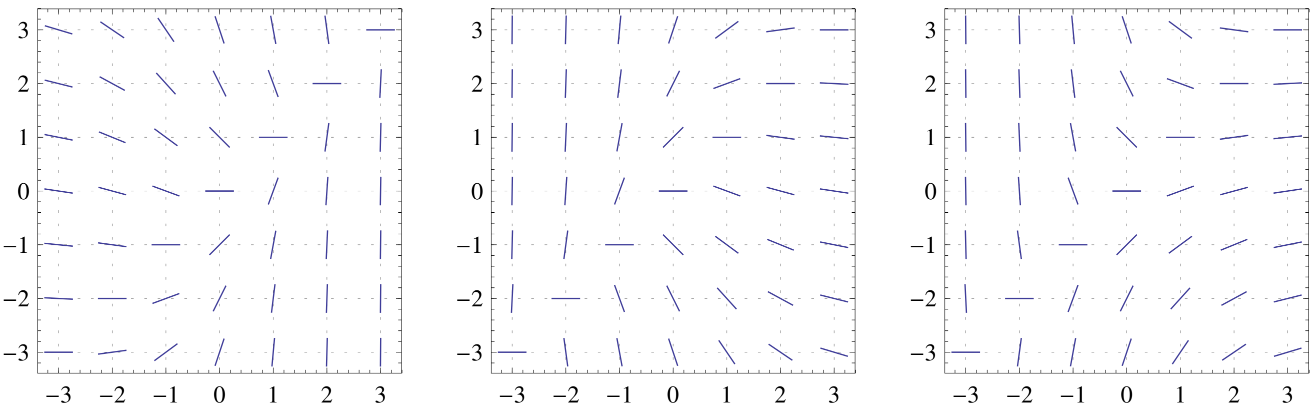
Problem 3. Circle the slope field below which belongs to the differential equation $e^x y' = x - y$.



Solution. A good point to carefully consider is $(1,3)$. By the DE, a solution passing through that point has slope y' satisfying $e^1\, y' = 1 - 3$. Equivalently, $y' = -2/e$. The only plot compatible with that is the third one.

Of course, we can arrive at the same conclusion based on other points or, even better, based on several points.

Problem 4. In the differential equation $x(y+1)\frac{\mathrm{d}y}{\mathrm{d}x} = (x^2+y)^3$ substitute $u = (x^2+y)^3$.

What is the resulting differential equation for u? No need to simplify!

Solution. If $u = (x^2+y)^3$, then $y = u^{1/3} - x^2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}u^{-2/3}\frac{\mathrm{d}u}{\mathrm{d}x} - 2x$.

Hence, the resulting differential equation for u is $x(u^{1/3} - x^2 + 1)\left(\frac{1}{3}u^{-2/3}\frac{\mathrm{d}u}{\mathrm{d}x} - 2x\right) = u$.

Problem 5. Solve the initial value problem $y' = 2xy + 3x^2 e^{x^2}$, $y(0) = 5$.

Solution. This is a linear DE. To solve it, we first bring it in the form $y' - 2xy = 3x^2 e^{x^2}$.

The integrating factor is $\exp(\int -2x\,\mathrm{d}x) = e^{-x^2}$.

We multiply the (rewritten) DE by e^{-x^2} to get $\underbrace{e^{-x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + -2xe^{-x^2}y}_{=\frac{\mathrm{d}}{\mathrm{d}x}[e^{-x^2}y]} = 3x^2$.

We then integrate both sides to get $e^{-x^2}y = x^3 + C$.

Using $y(0) = 5$, we find $5 = C$. Hence the solution is $y(x) = (x^3+5)e^{x^2}$.

Problem 6. Find a general solution to the differential equation $x(x+y)y' = y(3x+y)$.

Solution. This DE is neither separable nor linear. Hence, we look for a suitable substitution.

Since the right-hand side $3xy + y^2$ features a y^2, we divide both sides by x^2.

We get $\left(1 + \frac{y}{x}\right)y' = \frac{y}{x}\left(3 + \frac{y}{x}\right)$, which is homogeneous. We therefore substitute $u = \frac{y}{x}$. Then $y = ux$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u$.

The resulting DE is $(1+u)\left(x\frac{\mathrm{d}u}{\mathrm{d}x} + u\right) = u(3+u)$, which simplifies to $(1+u)x\frac{\mathrm{d}u}{\mathrm{d}x} = 2u$.

This DE is separable: $\frac{1+u}{u}\,\mathrm{d}u = \frac{2}{x}\,\mathrm{d}x$.

Integrating both sides, we find $u + \ln|u| = 2\ln|x| + C$.

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Exponentiating both sides simplifies this to $|u|e^u = e^{2\ln|x|+C} = e^C x^2$.

Assuming $u > 0$, we get $ue^u = Dx^2$ with $D = e^C > 0$. This implicit solution cannot be made more explicit using standard functions. For the original problem, we get the implicit solution $ye^{y/x} = Dx^3$.

Problem 7. Find a general solution to the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 \sin(x) = 0$.

Solution. This DE is separable:

$$\frac{\mathrm{d}y}{y^2} = -\sin(x)\,\mathrm{d}x \quad \Longrightarrow \quad \frac{-1}{y} = \cos(x) + C \quad \Longrightarrow \quad y = \frac{-1}{\cos(x) + C}.$$

In addition, there is the singular solution $y = 0$, which we lost when dividing by y^2.

Problem 8. Find a general solution to the differential equation $xy' = y + x^2 \cos(x)$.

Solution. This is a linear DE. To solve it, we first bring it in the form $y' - \frac{1}{x} y = x\cos(x)$.

The integrating factor is $e^{\int -\frac{1}{x}\mathrm{d}x} = e^{-\ln x} = \frac{1}{x}$.

We multiply the (rewritten) DE by $\frac{1}{x}$ to get $\underbrace{\frac{1}{x} y' - \frac{1}{x^2} y}_{= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{x} y\right]} = \cos(x)$.

We then integrate both sides to get $\frac{y}{x} = \sin(x) + C$.

Hence a general solution is $y = x\sin(x) + Cx$.

Problem 9. A tank contains 20gal of pure water. It is filled with brine (containing 5lb/gal salt) at a rate of 8gal/min. At the same time, well-mixed solution flows out at a rate of 6gal/min. How much salt is in the tank after t minutes?

Solution. Let $x(t)$ denote the amount of salt (in lb) in the tank after time t (in min).

At time t, the concentration of salt (in lb/gal) in the tank is $\frac{x(t)}{V(t)}$ where $V(t) = 20 + (8-6)t = 20 + 2t$ is the volume (in gal) in the tank.

In the time interval $[t, t + \Delta t]$: $\Delta x \approx 8 \cdot 5 \cdot \Delta t - 6 \cdot \frac{x(t)}{V(t)} \cdot \Delta t$.

Hence, $x(t)$ solves the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = 40 - \frac{6}{20 + 2t} x$ with $x(0) = 0$. Since this is a linear DE, we can solve it as follows:

- We write it in the form $\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{6}{20 + 2t} x = 40$.
- The integrating factor is $f(t) = \exp\left(\int \frac{6}{20 + 2t}\,\mathrm{d}t\right) = \exp(3\ln(20 + 2t)) = (20 + 2t)^3$.
- Multiply the (rewritten) DE by $f(t) = (20 + 2t)^3$ to get $\underbrace{(20 + 2t)^3 \frac{\mathrm{d}x}{\mathrm{d}t} + 6(20 + 2t)^2 x}_{= \frac{\mathrm{d}}{\mathrm{d}t}[(20+2t)^3 x]} = 40(20 + 2t)^3$.
- Integrate both sides to get $(20 + 2t)^3 x = 40 \int (20 + 2t)^3 \mathrm{d}t = 5(20 + 2t)^4 + C$.

Hence the general solution to the DE is $x(t) = 5(20 + 2t) + \frac{C}{(20 + 2t)^3}$. Using $x(0) = 0$, we find $100 + \frac{C}{20^3} = 0$ from which we conclude that $C = -100 \cdot 20^3 = -800{,}000$.

After t minutes, the tank therefore contains $x(t) = 5(20 + 2t) - \frac{800{,}000}{(20 + 2t)^3}$ pounds of salt.

Problem 10. The time rate of change of a rabbit population P is proportional to the square root of P. At time $t=0$, the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be after two months?

Solution. $P'(t)=k\sqrt{P}$ and $P(0)=100$, $P'(0)=20$. The problem asks for $P(2)$.

At $t=0$, we have $20=P'(0)=k\sqrt{P(0)}=10k$. Hence $k=2$.

The DE $\frac{\mathrm{d}P}{\mathrm{d}t}=2\sqrt{P}$ is separable: $P^{-1/2}\mathrm{d}P=2\mathrm{d}t$.

Integrating both sides, we find $2\sqrt{P}=2t+C$. From $P(0)=100$, we conclude that $C=20$.

Thus, $P(t)=(t+10)^2$. In particular, there will be $P(2)=12^2=144$ rabbits after two months.

Problem 11. A rising population is modeled by the differential equation $\frac{\mathrm{d}P}{\mathrm{d}t}=1000P-20P^2$.

(a) When the population size stabilizes in the long term, how big will the population be?

(b) Under which condition will the population size shrink?

(c) What is the population size when it is growing the fastest?

Solution.

(a) Once the population reaches a stable level in the long term, we have $\frac{\mathrm{d}P}{\mathrm{d}t}=0$ (no change in population size).

Hence, $0=1000P-20P^2=20P(50-P)$ which implies that $P=0$ or $P=50$. Since the population is rising, it will approach 50 in the long term.

(b) The population size will shrink if $\frac{\mathrm{d}P}{\mathrm{d}t}<0$.

The DE tells us that is the case if and only if $1000P-20P^2<0$ or, equivalently, if $P>\frac{1000}{20}=50$.

(c) This is asking when $\frac{\mathrm{d}P}{\mathrm{d}t}$ (the population growth) is maximal.

The DE is telling us that this growth is $f(P)=1000P-20P^2$. This a parabola that opens to the bottom. From Calculus, we know that it has a global maximum when $f'(P)=0$.

$f'(P)=1000-40P=0$ leads to $P=25$.

Thus, the population is growing the fastest when its size is 25.

Comment. You probably noticed that the DE is the logistic equation. The first problem therefore is asking about the carrying capacity which we could determine by matching parameters. However, note that we are able to (easily!) decide all the above questions directly from the DE without needing the solution of the logistic equation.

Problem 12. Solve the initial value problem $x\frac{\mathrm{d}y}{\mathrm{d}x}=y-xe^{y/x}, \quad y(1)=0$.

Solution. This DE is neither separable nor linear. Hence, we look for a suitable substitution.

Since the right-hand side features a y, we divide both sides by x. We get $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{x}-e^{y/x}$ which is homogeneous.

We therefore substitute $u=\frac{y}{x}$. Then $y=ux$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=x\frac{\mathrm{d}u}{\mathrm{d}x}+u$.

The resulting DE is $x\frac{\mathrm{d}u}{\mathrm{d}x}+u=u-e^u$, which simplifies to $x\frac{\mathrm{d}u}{\mathrm{d}x}=-e^u$.

This DE is separable: $e^{-u}\,\mathrm{d}u=-\frac{1}{x}\,\mathrm{d}x$. Integrating both sides, we find $-e^{-u}=-\ln|x|+C$.

Accordingly, for the initial DE, $-e^{-y/x}=-\ln(x)+C$. (Note that $x>0$, at least locally, due to the initial condition.)

Using $y(1)=0$ we find $-e^{0/1}=-\ln(1)+C$ so that $C=-1$.

Thus $e^{-y/x}=\ln(x)+1$ and, therefore, $y=-x\ln(\ln(x)+1)$.

Problem 13. Solve the initial value problem $(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x}+xy=\frac{1}{\sqrt{x^2+1}}, \quad y(0)=1.$

Solution. This is a linear DE. To solve it, we first bring it in the form $\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{x}{x^2+1}y=\frac{1}{(x^2+1)^{3/2}}$.

The integrating factor is $\exp\left(\int \frac{x}{x^2+1}\mathrm{d}x\right)=\exp\left(\frac{1}{2}\ln(x^2+1)\right)=(x^2+1)^{1/2}$.

We multiply the (rewritten) DE by $(x^2+1)^{1/2}$ to get $\underbrace{(x^2+1)^{1/2}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{x}{(x^2+1)^{1/2}}y}_{=\frac{\mathrm{d}}{\mathrm{d}x}[(x^2+1)^{1/2}y]}=\frac{1}{x^2+1}$.

We then integrate both sides to get $(x^2+1)^{1/2}y=\arctan(x)+C$.

Since $y(0)=1$, we find $C=1$. Therefore,

$$y(x)=\frac{\arctan(x)+1}{(x^2+1)^{1/2}}.$$

Problem 14. Find a general solution to the differential equation $2+\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{2x+y}$.

Solution. This DE is neither separable nor linear. Hence, we look for a suitable substitution.

Note that this DE is of the form $y'=F(2x+y)$ with $F(t)=\sqrt{t}-2$.

We therefore substitute $u=2x+y$. Then $y=u-2x$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}u}{\mathrm{d}x}-2$.

The resulting DE is $2+\left(\frac{\mathrm{d}u}{\mathrm{d}x}-2\right)=\sqrt{u}$ or, simplified, $\frac{\mathrm{d}u}{\mathrm{d}x}=\sqrt{u}$.

This DE is separable: $u^{-1/2}\mathrm{d}u=\mathrm{d}x$. Integrating both sides, we find $2u^{1/2}=x+C$.

Hence $u=\frac{1}{4}(x+C)^2$ and $y=u-2x=\frac{1}{4}(x+C)^2-2x$ [which is a solution as long as $x+C>0$].

Problem 15. Find a general solution to the differential equation $x^2\frac{\mathrm{d}y}{\mathrm{d}x}-x^2-y^2-3xy=0$.

Solution. This DE is neither separable nor linear. Hence, we look for a suitable substitution.

Since the left-hand side features a y^2, we divide both sides by x^2. We get $\frac{\mathrm{d}y}{\mathrm{d}x}-1-(\frac{y}{x})^2-3\frac{y}{x}=0$ which is homogeneous.

We therefore substitute $u=\frac{y}{x}$. Then $y=ux$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=x\frac{\mathrm{d}u}{\mathrm{d}x}+u$.

The resulting DE is $\left(x\frac{\mathrm{d}u}{\mathrm{d}x}+u\right)-1-u^2-3u=0$, which simplifies to $x\frac{\mathrm{d}u}{\mathrm{d}x}=u^2+2u+1=(u+1)^2$.

This DE is separable: $\frac{1}{(u+1)^2}\mathrm{d}u=\frac{1}{x}\mathrm{d}x$. Integrating both sides, we find $-\frac{1}{u+1}=\ln|x|+C$.

Hence $u+1=-\frac{1}{\ln|x|+C}$ and thus, for the original DE,

$$y(x)=ux=-\frac{x}{\ln|x|+C}-x.$$

Comment. When we divided by $u+1$, we lost the singular solution $u=-1$ which corresponds to the solution $y=-x$ of the original DE.

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