Review: Computing derivatives

Given a function $y(x)$, we learned in Calculus I that its **derivative**

$$y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

(where $\Delta y = y(x + \Delta x) - y(x)$) has the following two important characterizations:

- $y'(x)$ is the **slope of the tangent line** of the graph of $y(x)$ at x, and
- $y'(x)$ is the **rate of change** of $y(x)$ at x.

Moreover, we learned simple rules to compute the derivative of functions:

- **(sum rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)] = f'(x) + g'(x)$
- **(product rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
- **(chain rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x)$

  Comment. If we write $t = g(x)$ and $y = f(t)$, then the chain rule takes the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x}$.
  In other words, the chain rule expresses the fact that we can treat $\frac{\mathrm{d}y}{\mathrm{d}x}$ (which initially is just a notation for $y'(x)$) as an honest fraction.

- **(basic functions)** $\frac{\mathrm{d}}{\mathrm{d}x} x^r = r x^{r-1}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$, $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} \sin(x) = \cos(x)$, $\frac{\mathrm{d}}{\mathrm{d}x} \cos(x) = -\sin(x)$

These rules are enough to compute the derivative of any function that we can build from the basic functions using algebraic operations and composition. On the other hand, as you probably recall from Calculus II, reversing the operation of differentiation (i.e. computing antiderivatives) is much more difficult.

In particular, there exist simple functions (such as e^{x^2}) whose antiderivative cannot be expressed in terms of the basic functions above.

Example 1. Derive the **quotient rule** from the rules above.

Solution. We write $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$ and apply the product rule to get

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x) \frac{1}{g(x)} + f(x) \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{g(x)}.$$

By the chain rule combined with $\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{x} = -\frac{1}{x^2}$, we have $\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{g(x)} = -\frac{1}{g(x)^2} g'(x)$. Using this in the previous formula,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x) \frac{1}{g(x)} - f(x) \frac{1}{g(x)^2} g'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2}.$$

Putting the final two fractions on a common denominator, we obtain the familiar quotient rule

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

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First examples of differential equations

Example 2. If $y(x) = e^{x^2}$ then $y'(x) = 2xe^{x^2} = 2xy(x)$ or, for short, $y' = 2xy$.

Accordingly, we say that $y(x) = e^{x^2}$ is a **solution** to the **differential equation** (DE) $y' = 2xy$.

Example 3. By computing its derivative, determine a DE solved by $y(x) = \sin(3x)$.

Solution. $y'(x) = 3\cos(3x) = 3\sqrt{1 - (\sin(3x))^2} = 3\sqrt{1 - y(x)^2}$

[Here we used that $\cos(x)^2 + \sin(x)^2 = 1$, which implies that $\cos(x) = \sqrt{1 - \sin(x)^2}$.]

Hence, $y(x) = \sin(3x)$ solves the differential equation $y' = 3\sqrt{1 - y^2}$.

Example 4. By computing its second derivative, determine another DE solved by $y(x) = \sin(3x)$.

Solution. $y''(x) = -9\sin(3x) = -9y(x)$.
Thus, $y(x) = \sin(3x)$ also solves the differential equation $y'' = -9y$.

If the highest derivative appearing in a DE is an rth derivative, we say that the DE has **order** r.

For instance. The DE $y' = 3\sqrt{1 - y^2}$ has order 1 (such DEs are also called first order DEs).
On the other hand, the DE $y'' = -9y$ has order 2 (such DEs are also called second order DEs).

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Example 5. Verify that $e^y y' = 1$ is solved by $y(x) = \ln(x + C)$.

Solution. $y'(x) = \frac{1}{x + C}$ and $e^{y(x)} = x + C$.

Hence, $e^y y' = (x + C)\frac{1}{x + C} = 1$.

Because $y(x)$ solves the DE for any value of the parameter C, we say that $y(x) = \ln(x + C)$ is a **one-parameter family** of solutions to the DE.

Example 6. Consider the DE $y'' = y' + 6y$. For which r is e^{rx} a solution?

Solution. If $y(x) = e^{rx}$, then $y'(x) = re^{rx}$ and $y''(x) = r^2 e^{rx}$.

Plugging $y(x) = e^{rx}$ into the DE, we get $r^2 e^{rx} = re^{rx} + 6e^{rx}$ which simplifies to $r^2 = r + 6$.

This has the two solutions $r = -2$, $r = 3$. Hence e^{-2x} and e^{3x} are solutions of the DE.

In fact, we check that $Ae^{-2x} + Be^{3x}$ is a **two-parameter family** of solutions to the DE.

Important comment. It is no coincidence that the order of the DE is 2, whereas the previous example has order 1. In general, we expect a DE of order r to have a solution with r parameters.

Example 7. Solve the DE $y' = x^2 + x$.

Solution. Note that the DE simply asks for a function $y(x)$ with a specific derivative (in particular, the right-hand side does not involve $y(x)$). In other words, the desired $y(x)$ is an **antiderivative** of $x^2 + x$. We know from Calculus II that we can find antiderivatives by integrating:

$$y(x) = \int (x^2 + x)\mathrm{d}x = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

Moreover, we know from Calculus II that there are no other solutions. In other words, we found the **general solution** to the DE.

To single out a **particular solution**, we need to specify additional conditions (typically one condition per parameter in the general solution). For instance, it is common to impose **initial conditions** such as $y(1) = 2$. A DE together with an initial condition is called an **initial value problem** (IVP).

Example 8. Solve the IVP $y' = x^2 + x$ with $y(1) = 2$.

Solution. From the previous example, we know that $y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$.

Since $y(1) = \frac{1}{3} + \frac{1}{2} + C = \frac{5}{6} + C \overset{!}{=} 2$, we find $C = 2 - \frac{5}{6} = \frac{7}{6}$.

Hence, $y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{7}{6}$ is the (unique) solution of the IVP.

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Example 9. Solve the DE $y'' = x^2 + x$.

Solution. We now take two antiderivatives of $x^2 + x$ to get

$$y(x) = \int\int (x^2 + x)\mathrm{d}x\mathrm{d}x = \int \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + C\right)\mathrm{d}x = \frac{1}{12}x^4 + \frac{1}{6}x^3 + Cx + D,$$

where it is important that we give the second constant of integration a name different from the first.

Again, this is the **general solution** to the DE. The DE is of order 2 and, as expected, the general solution has 2 parameters.

Important. Note that we are working with functions $y(x)$ of a single variable. This allows us to write simply y' for $\frac{\mathrm{d}}{\mathrm{d}x}y(x)$ without risk of confusion.

Of course, we may use different variables such as $x(t)$ and $x' = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, as long as this is clear from the context.

Differential equations that involve only derivatives with respect to a single variable are known as **ordinary differential equations** (ODEs).

On the other hand, differential equations that involve derivatives with respect to several variables are referred to as **partial differential equations** (PDEs).

Example 10. The DE

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 u(x,y) + \left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^2 u(x,y) = 0,$$

often abbreviated as $u_{xx} + u_{yy} = 0$, is a partial differential equation in two variables.

This particular PDE is known as **Laplace's equation** and describes, for instance, steady-state heat distributions.
https://en.wikipedia.org/wiki/Laplace%27s_equation

This and other fundamental PDEs will be discussed in Differential Equations II.

Example 11. (review)

(a) Verify that $x(t) = \frac{1}{c - kt}$ is a one-parameter family of solutions to the DE $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$.

(b) Solve the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$, $x(0) = 2$.

(c) Solve the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$, $x(0) = 0$.

Solution.

(a) We compute that $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{(c - kt)^2} \cdot (-k) = \frac{k}{(c - kt)^2}$.

On the other hand, $kx^2 = k\left(\frac{1}{c - kt}\right)^2 = \frac{k}{(c - kt)^2}$ as well. Thus, indeed, $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$.

(b) We start with $x(t) = \frac{1}{c - kt}$ (which we know solves the DE for any value of c) and seek to choose c so that $x(0) = 2$.

Since $x(0) = \left[\frac{1}{c - kt}\right]_{t=0} = \frac{1}{c} \overset{!}{=} 2$, we find $c = \frac{1}{2}$.

Hence, the IVP has the (unique) solution $x(t) = \frac{1}{1/2 - kt}$.

(c) Proceeding as in the previous part, we now arrive at the impossible equation $\frac{1}{c} \overset{!}{=} 0$.

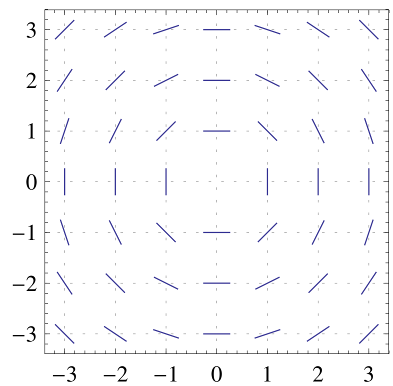
However, this suggests that we should consider taking $c \to \infty$ in $x(t) = \frac{1}{c - kt}$, which results in $x(t) = 0$.

Indeed, it is easy to verify (make sure you know what this entails!) that $x(t) = 0$ solves the IVP.

Slope fields, or sketching solutions to DEs

Example 12. Consider the DE $y' = -x/y$.

Let's pick a point, say, $(1, 2)$. If a solution $y(x)$ is passing through that point, then its slope has to be $y' = -1/2$. We therefore draw a small line through the point $(1, 2)$ with slope $-1/2$. Continuing in this fashion for several other points, we obtain the **slope field** on the right.



With just a little bit of imagination, we can now anticipate the solutions to look like (half)circles around the origin. Let us check whether $y(x) = \sqrt{r^2 - x^2}$ might indeed be a solution!

$y'(x) = \frac{1}{2} \frac{-2x}{\sqrt{r^2 - x^2}} = -x/y(x)$. So, yes, we actually found solutions!

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Solving DEs: Separation of variables

Example 13. Solve the DE $y' = -\frac{x}{y}$.

Solution. Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

Separate the variables to get $y\,\mathrm{d}y = -x\,\mathrm{d}x$ (in particular, we are multiplying both sides by $\mathrm{d}x$).

Integrating both sides, we get $\int y\,\mathrm{d}y = \int -x\,\mathrm{d}x$.

Computing both integrals results in $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$ (we combine the two constants of integration into one).

Hence $x^2 + y^2 = D$ (with $D = 2C$).

This is an **implicit form** of the solutions to the DE. We can make it explicit by solving for y. Doing so, we find $y(x) = \pm\sqrt{D - x^2}$ (choosing $+$ gives us the upper half of a circle, while the negative sign gives us the lower half).

Comment. The step above where we break $\frac{\mathrm{d}y}{\mathrm{d}x}$ apart and then integrate may sound sketchy!

However, keep in mind that, after we find a solution $y(x)$, even if by sketchy means, we can (and should!) verify that $y(x)$ is indeed a solution by plugging into the DE. We actually already did that in the previous example!

In general, **separation of variables** solves $y' = g(x)h(y)$ by writing the DE as $\frac{1}{h(y)}\mathrm{d}y = g(x)\mathrm{d}x$.

Note that $\frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$ is indeed equivalent to $\int \frac{1}{h(y)}\mathrm{d}y = \int g(x)\mathrm{d}x + C$. Why?! (Apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to the integrals...)

Example 14. Solve the IVP $y' = -\frac{x}{y}$, $y(0) = -3$.

Comment. Instead of using what we found in the previous example, we start from scratch to better illustrate the solution process (and how we can use the initial condition right away to determine the value of the constant of integration).

Solution. We separate variables to get $y\,\mathrm{d}y = -x\,\mathrm{d}x$.

Integrating gives $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$, and we use $y(0) = -3$ to find $\frac{1}{2}(-3)^2 = 0 + C$ so that $C = \frac{9}{2}$.

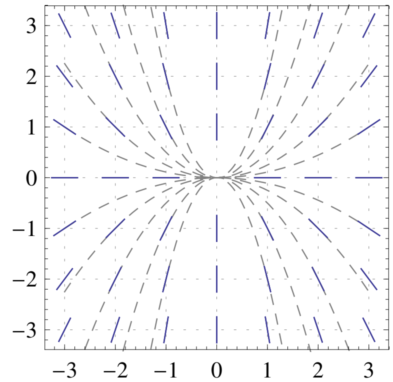
Hence, $x^2 + y^2 = 9$ is an **implicit** form of the solution.

Solving for y, we get $y = -\sqrt{9 - x^2}$ (note that we have to choose the negative sign so that $y(0) = -3$).

Comment. Note that our solution is a **local solution**, meaning that it is valid (and solves the DE) locally around $x = 0$ (from the initial condition). However, it is not a **global solution** because it doesn't make sense outside of x in the interval $[-3, 3]$.

Example 15. Consider the DE $xy' = 2y$.

Sketch its slope field.



Challenge. Try to guess solutions $y(x)$ from the slope field.

Solution. For instance, to find the slope at the point $(3,1)$, we plug $x=3$, $y=1$ into the DE to get $3y' = 2$. Hence, the slope is $y' = 2/3$.

The resulting slope field is sketched on the right.

Solution of the challenge. Trace out the solution through $(1,1)$ (and then some other points). Their shape looks like a parabola, so that we might guess that $y(x) = Cx^2$ solves the DE. Check that this is indeed the case by plugging into the DE!

Example 16. Solve the IVP $xy' = 2y$, $y(1) = 2$.

Solution. Rewrite the DE as $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x}$.

Then multiply both sides with $\mathrm{d}x$ and integrate both of them to get $\int \frac{1}{y}\mathrm{d}y = \int \frac{2}{x}\mathrm{d}x$.

Hence, $\ln|y| = 2\ln|x| + C$.

The initial condition $y(1) = 2$ tells us that, at least locally, $x > 0$ and $y > 0$. Thus $\ln(y) = 2\ln(x) + C$.

Moreover, plugging in $x = 1$ and $y = 2$, we find $C = \ln(2)$.

Solving $\ln(y) = 2\ln(x) + \ln(2)$ for y, we find $y = e^{2\ln(x) + \ln(2)} = 2x^2$.

Comment. When solving a DE or IVP, we can generally only expect to find a **local solution**, meaning that our solution might only be valid in a small interval around the initial condition (here, we can only expect $y(x)$ to be a solution for all x in an interval around 1; especially since we assumed $x > 0$ in our solution). However, we can check (do it!) that the solution $y = 2x^2$ is actually a **global solution** (meaning that it is a solution for all x, not just locally around 1).

Example 17. Solve the IVP $xy' = 2y$, $y(1) = -1$.

Solution. Again, we rewrite the DE as $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x}$, multiply both sides with $\mathrm{d}x$, and integrate to get $\int \frac{1}{y}\mathrm{d}y = \int \frac{2}{x}\mathrm{d}x$.

Hence, $\ln|y| = 2\ln|x| + C$. The initial condition $y(1) = -1$ tells us that, at least locally, $x > 0$ and $y < 0$ (note that this means $|y| = -y$). Thus $\ln(-y) = 2\ln(x) + C$.

Moreover, plugging in $x = 1$ and $y = -1$, we find $C = 0$.

Solving $\ln(-y) = 2\ln(x)$ for y, we find $y = -e^{2\ln(x)} = -x^2$. We easily verify that this is indeed a global solution.

Example 18. $y' = x + y$ is a DE for which the variables cannot be separated.

No worries, very soon we will have several tools to solve this DE as well.

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Existence and uniqueness of solutions

The following is a very general result that allows us to guarantee that "nice" IVPs must have a solution and that this solution is unique.

Comment. Note that any first-order DE can be written as $g(y', y, x) = 0$ where g is some function of three variables. Assuming that g is reasonable, we can solve for y' and rewrite such a DE as $y' = f(x, y)$ (for some, possibly complicated, function f).

Comment. To be precise, a solution to the IVP $y' = f(x, y)$, $y(a) = b$ is a function $y(x)$, defined on an interval I containing a, such that $y'(x) = f(x, y(x))$ for all $x \in I$ and $y(a) = b$.

Theorem 19. (existence and uniqueness) Consider the IVP $y' = f(x, y)$, $y(a) = b$.
If both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous [in a rectangle] around (a, b), then the IVP has a unique solution in some interval $x \in (a - \delta, a + \delta)$ where $\delta > 0$.

Comment. The interval around a might be very small. In other words, the δ in the theorem could be very small.

Comment. Note that the theorem makes two important assertions. First, it says that there exists a **local solution**. Second, it says that this solution is unique. These two parts of the theorem are famous results usually attributed to Peano (existence) and Picard–Lindelöf (uniqueness).

Advanced comment. The condition about $\frac{\partial}{\partial y} f(x, y)$ is a bit technical (and not optimal). If we drop this condition, we still get existence but, in general, no longer uniqueness.

Advanced comment. The interval in which the solution is unique could be smaller than the interval in which it exists. In other words, it is possible that, away from the initial condition, the solution "forks" into two or more solutions. Note that this does not contradict the theorem because it only guarantees uniqueness on a small interval.

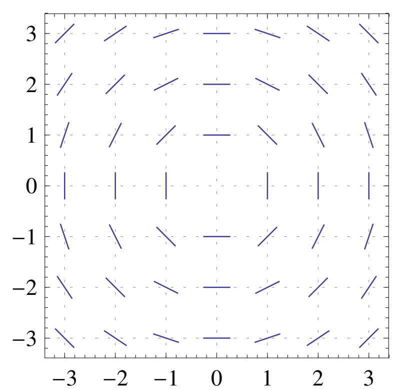
Example 20. Consider, again, the IVP $y' = -x/y$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = -x/y$.

We compute that $\frac{\partial}{\partial y} f(x, y) = x/y^2$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $y \neq 0$.

Hence, if $b \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.



Comment. In Example 13, we found that the DE $y' = -x/y$ is solved by $y(x) = \pm\sqrt{D - x^2}$.

Assume $b > 0$ (things work similarly for $b < 0$). Then $y(x) = \sqrt{D - x^2}$ solves the IVP (we need to choose D so that $y(a) = b$) if we choose $D = a^2 + b^2$. This confirms that there exists a solution. On the other hand, uniqueness means that there can be no other solution to the IVP than this one.

What happens in the case $b = 0$?

Solution. In this case, the existence and uniqueness theorem does not guarantee anything. If $a \neq 0$, then $y(x) = \sqrt{a^2 - x^2}$ and $y(x) = -\sqrt{a^2 - x^2}$ both solve the IVP (so we certainly don't have uniqueness), however only in a weak sense: namely, both of these solutions are not valid locally around $x = a$ but only in an interval of which a is an endpoint (for instance, the IVP $y' = -x/y$, $y(2) = 0$ is solved by $y(x) = \pm\sqrt{4 - x^2}$ but both of these solutions are only valid on the interval $[-2, 2]$ which ends at 2, and neither of these solutions can be extended past 2).

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Review. Existence and uniqueness theorem (Theorem 19) for an IVP $y' = f(x, y)$, $y(a) = b$:
If $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous around (a, b) then, locally, the IVP has a unique solution.

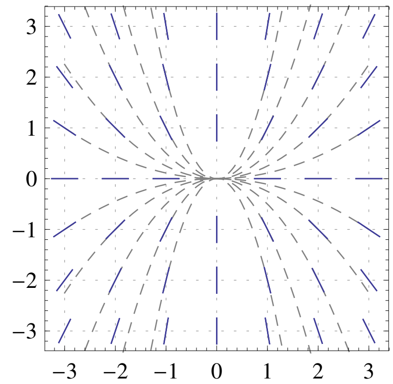
Example 21. Consider, again, the IVP $xy' = 2y$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = 2y/x$.

We compute that $\frac{\partial}{\partial y} f(x, y) = 2/x$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $x \neq 0$.

Hence, if $a \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.



What happens in the case $a = 0$?

Solution. In Example 15, we found that the DE $xy' = 2y$ is solved by $y(x) = Cx^2$.

This means that the IVP with $y(0) = 0$ has infinitely many solutions.

On the other hand, the IVP with $y(0) = b$ where $b \neq 0$ has no solutions. (This follows from the fact that there are no solutions to the DE besides $y(x) = Cx^2$. Can you see this by looking at the slope field?)

Example 22. Consider the IVP $y' = ky^2$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = ky^2$. We compute that $\frac{\partial}{\partial y} f(x, y) = 2ky$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y).

Hence, for any initial conditions, the IVP locally has a unique solution by the existence and uniqueness theorem.

Example 23. Solve $y' = ky^2$.

Solution. Separate variables to get $\frac{1}{y^2} \frac{\mathrm{d}y}{\mathrm{d}x} = k$.

Integrating $\int \frac{1}{y^2} \mathrm{d}y = \int k \mathrm{d}x$, we find $-\frac{1}{y} = kx + C$.

We solve for y to get $y = -\frac{1}{C + kx} = \frac{1}{D - kx}$ (with $D = -C$). That is the solution we verified earlier!

Comment. Note that we did not find the solution $y = 0$ (it was "lost" when we divided by y^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). However, note that we can obtain it from the general solution by letting $D \to \infty$.

Caution. We have to be careful about transforming our DE when using separation of variables: Just as the division by y^2 made us lose a solution, other transformations can add extra solutions which do not solve the original DE.

Here is a silly example (silly, because the transformation serves no purpose here) which still illustrates the point. The DE $(y-1)y' = (y-1)ky^2$ has the same solutions as $y' = ky^2$ plus the additional solution $y = 1$ (which does not solve $y' = ky^2$).

Example 24. (extra) Solve the IVP $y' = y^2$, $y(0) = 1$.

Solution. From the previous example with $k = 1$, we know that $y(x) = \frac{1}{D - x}$.

Using $y(0) = 1$, we find that $D = 1$ so that the unique solution to the IVP is $y(x) = \frac{1}{1 - x}$.

Comment. Note that we already concluded the uniqueness from the existence and uniqueness theorem.

On the other hand, note that $y(x) = \frac{1}{1 - x}$ is only valid on $(-\infty, 1)$ and that it cannot be continuously extended past $x = 1$; it is only a local solution.

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Example 25. (homework) Consider the IVP $(x-y^2)y'=3x$, $y(4)=b$. For which choices of b does the existence and uniqueness theorem guarantee a unique (local) solution?

Solution. The IVP is $y'=f(x,y)$ with $f(x,y)=3x/(x-y^2)$. We compute that $\frac{\partial}{\partial y}f(x,y)=6xy/(x-y^2)^2$.

We observe that both $f(x,y)$ and $\frac{\partial}{\partial y}f(x,y)$ are continuous for all (x,y) with $x-y^2\neq 0$.

Note that $4-b^2\neq 0$ is equivalent to $b\neq\pm 2$.

Hence, if $b\neq\pm 2$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Linear first-order DEs

A **linear differential equation** is one where the function y and its derivatives only show up linearly (i.e. there are no terms such as y^2, $1/y$ or $\sin(y)$).

As such, the most general linear first-order DE is of the form

$$A(x)y'+B(x)y+C(x)=0.$$

Comment. Note that any such DE can also be rewritten in the form $y'+P(x)y=Q(x)$ by dividing by $A(x)$ and rearranging. We will use this form when solving linear first-order DEs.

Example 26. (extra) Solve $\frac{\mathrm{d}y}{\mathrm{d}x}=2xy^2$.

Solution. (separation of variables) $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}=2x$, $-\frac{1}{y}=x^2+C$.

Hence the general solution is $y=\frac{1}{D-x^2}$. [There also is the singular solution $y=0$.]

Solution. (in other words) Note that $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}=2x$ can be written as $\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{1}{y}\right]=\frac{\mathrm{d}}{\mathrm{d}x}[x^2]$.

From there it follows that $-\frac{1}{y}=x^2+C$, as above.

We now use the idea of writing both sides as a derivative to also solve linear DEs that are not separable.

The multiplication by $\frac{1}{y^2}$ will be replaced by multiplication with a so-called **integrating factor**.

Example 27. Solve $y'=x-y$.

Comment. Note that we cannot use separation of variables this time.

Solution. Rewrite the DE as $y'+y=x$.

Next, multiply both sides with e^x (we will see in a little bit how to find this "integrating factor") to get

$$\underbrace{e^x y'+e^x y}_{=\frac{\mathrm{d}}{\mathrm{d}x}[e^x y]}=xe^x.$$

The "magic" part is that we are able to realize the new left-hand side as a derivative!

Next, we will integrate both sides and then solve for y. (Try it yourself!) To be continued…

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Example 28. (resumed) Solve $y' = x - y$.

Comment. Note that we cannot use separation of variables this time.

Solution. Rewrite the DE as $y' + y = x$.
Next, multiply both sides with e^x (we will see in a little bit how to find this "integrating factor") to get

$$\underbrace{e^x y' + e^x y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[e^x y]} = x e^x.$$

The "magic" part is that we are able to realize the new left-hand side as a derivative!
We can then integrate both sides to get

$$e^x y = \int x e^x \mathrm{d}x = x e^x - e^x + C.$$

From here it follows that $y = x - 1 + C e^{-x}$.

Comment. For the final integral, we used that $\int x e^x \mathrm{d}x = x e^x - \int e^x \mathrm{d}x = x e^x - e^x + C$ which follows, for instance, via integration by parts (with $f(x) = x$ and $g'(x) = e^x$ in the formula reviewed below).

Review. The multiplication rule $(fg)' = f' g + f g'$ implies $fg = \int f' g + \int f g'$.
The latter is equivalent to **integration by parts**:

$$\int f(x) g'(x) \mathrm{d}x = f(x) g(x) - \int f'(x) g(x) \mathrm{d}x$$

Comment. Sometimes, one writes $g'(x)\mathrm{d}x = \mathrm{d}g(x)$.

In general, we can solve any **linear first-order DE** $y' + P(x) y = Q(x)$ in this way.

- We want to multiply with an **integrating factor** $f(x)$ such that the left-hand side of the DE becomes

  $$f(x) y' + f(x) P(x) y = \frac{\mathrm{d}}{\mathrm{d}x}[f(x) y].$$

  Since $\frac{\mathrm{d}}{\mathrm{d}x}[f(x) y] = f(x) y' + f'(x) y$, we need $f'(x) = f(x) P(x)$ for that.

- Check that $f(x) = \exp\left(\int P(x) \mathrm{d}x\right)$ has this property.

  Comment. This follows directly from computing the derivative of this $f(x)$ via the chain rule.

  Homework. On the other hand, note that finding f meant solving the DE $f' = P(x)\, f$. This is a separable DE. Solve it by separation of variables to arrive at the above formula for $f(x)$ yourself.

  Just to make sure. There is no difference between $\exp(x)$ and e^x. Here, we prefer the former notation for typographical reasons.

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With that integrating factor, we have the following recipe for solving any linear first-order equation:

(solving linear first-order DEs)

(a) Write the DE in the form $y' + P(x)y = Q(x)$.

(b) Compute the integrating factor as $f(x) = \exp\left(\int P(x)\mathrm{d}x\right)$.

[We can choose any constant of integration.]

(c) Multiply the DE from part (a) by $f(x)$ to get

$$\underbrace{f(x)y' + f(x)P(x)y}_{=\frac{\mathrm{d}}{\mathrm{d}x}[f(x)y]} = f(x)Q(x).$$

(d) Integrate both sides to get

$$f(x)y = \int f(x)Q(x)\mathrm{d}x + C.$$

Then solve for y by dividing by $f(x)$.

Comment. For better understanding, we prefer to go through the above steps. On the other hand, we can combine these steps into the following formula for the general solution of $y' + P(x)y = Q(x)$:

$$y = \frac{1}{f(x)}\left(\int f(x)Q(x)\mathrm{d}x + C\right) \quad \text{where } f(x) = e^{\int P(x)\mathrm{d}x}$$

Existence and uniqueness. Note that the solution we construct exists on any interval on which P and Q are continuous (not just on some possibly very small interval). This is better than what the existence and uniqueness theorem (Theorem 19) can guarantee. This is one of the many ways in which linear DEs have particularly nice properties compared to DEs in general.

Example 29. Solve $x^2 y' = 1 - xy + 2x$, $y(1) = 3$.

Solution. This is a linear first-order DE. We can therefore solve it according to the recipe above.

(a) Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ with $P(x) = \frac{1}{x}$ and $Q(x) = \frac{1}{x^2} + \frac{2}{x}$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^{\ln x} = x$.

Here, we could write $\ln x$ instead of $\ln|x|$ because the initial condition tells us that $x > 0$, at least locally.

Comment. We can also choose a different constant of integration but that would only complicate things.

(c) Multiply the (rewritten) DE by $f(x) = x$ to get

$$\underbrace{x\frac{\mathrm{d}y}{\mathrm{d}x} + y}_{=\frac{\mathrm{d}}{\mathrm{d}x}[xy]} = \frac{1}{x} + 2.$$

(d) Integrate both sides to get (again, we use that $x > 0$ to avoid having to use $|x|$)

$$xy = \int\left(\frac{1}{x} + 2\right)\mathrm{d}x = \ln x + 2x + C.$$

Using $y(1) = 3$ to find C, we get $1 \cdot 3 = \ln(1) + 2 \cdot 1 + C$ which results in $C = 3 - 2 = 1$.

Hence, the (unique) solution to the IVP is $y = \frac{\ln(x) + 2x + 1}{x}$.

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Example 30. (extra) Solve $y' = 2y + 3x - 1$, $y(0) = 2$.

Solution. This is a linear first-order DE.

(a) Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ with $P(x) = -2$ and $Q(x) = 3x - 1$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^{-2x}$.

(c) Multiply the (rewritten) DE by $f(x) = e^{-2x}$ to get

$$\underbrace{e^{-2x}\frac{\mathrm{d}y}{\mathrm{d}x} - 2e^{-2x}y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[e^{-2x}y]} = (3x-1)e^{-2x}.$$

(d) Integrate both sides to get

$$\begin{aligned} e^{-2x}y &= \int (3x-1)e^{-2x}\mathrm{d}x \\ &= 3\int x e^{-2x}\mathrm{d}x - \int e^{-2x}\mathrm{d}x \\ &= 3\left(-\frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x}\right) - \left(-\frac{1}{2}e^{-2x}\right) + C \\ &= -\frac{3}{2}x e^{-2x} - \frac{1}{4}e^{-2x} + C. \end{aligned}$$

Here, we used that $\int x e^{-2x}\mathrm{d}x = -\frac{1}{2}x e^{-2x} + \frac{1}{2}\int e^{-2x}\mathrm{d}x = -\frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x}$ (for instance, via integration by parts with $f(x) = x$ and $g'(x) = e^{-2x}$).

Hence, the general solution is $y(x) = -\frac{3}{2}x - \frac{1}{4} + Ce^{2x}$.

Solving $y(0) = -\frac{1}{4} + C = 2$ for C yields $C = \frac{9}{4}$.

In conclusion, the (unique) solution to the IVP is $y(x) = -\frac{3}{2}x - \frac{1}{4} + \frac{9}{4}e^{2x}$.

Substitutions in DEs

Example 31. (review) Using substitution, compute $\int \frac{x}{1+x^2}\mathrm{d}x$.

Solution. We substitute $t = 1 + x^2$. In that case, $\mathrm{d}t = 2x\,\mathrm{d}x$.
$\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\int \frac{1}{t}\mathrm{d}t = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln(1+x^2) + C$

Comment. Why were we allowed to drop the absolute value in the logarithm?

Review. On the other hand, recall that $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan(x) + C$.

Example 32. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = (x+y)^2$.

First things first. Is this DE separable? Is it linear? (No to both but make sure that this is clear to you.) This means that our previous techniques are not sufficient to solve this DE.

Solution. Looking at the right-hand side, we have a feeling that the substitution $u = x + y$ might simplify things. Then $y = u - x$ and, therefore, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} - 1$.

Using these, the DE translates into $\frac{\mathrm{d}u}{\mathrm{d}x} - 1 = u^2$. This is a separable DE: $\frac{1}{1+u^2}\mathrm{d}u = \mathrm{d}x$

After integration, we find $\arctan(u) = x + C$ and, thus, $u = \tan(x + C)$.

The solution of the original DE is $y = u - x = \tan(x + C) - x$.

Example 33. Consider the DE $x\frac{\mathrm{d}y}{\mathrm{d}x} = y + y^2 f(x)$.

(a) Substitute $u = \frac{y}{x}$. Is the resulting DE separable or linear?

(b) Substitute $v = \frac{1}{y}$. Is the resulting DE separable or linear?

(c) **(homework)** Solve each of the new DEs.

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Solution.

(a) Set $u = \frac{y}{x}$. Then $y = ux$ and, thus, $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u$.

Using these, the DE translates into $x\left(x\frac{\mathrm{d}u}{\mathrm{d}x} + u\right) = ux + (ux)^2 f(x)$.

This DE simplifies to $\frac{\mathrm{d}u}{\mathrm{d}x} = u^2 f(x)$. This is a separable DE.

(b) Set $v = \frac{1}{y}$. Then $y = \frac{1}{v}$ and, thus, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x}$.

Using these, the DE translates into $x\left(-\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x}\right) = \frac{1}{v} + \frac{1}{v^2} f(x)$.

This DE simplifies to $x\frac{\mathrm{d}v}{\mathrm{d}x} = -v - f(x)$. This is a linear DE.

(c) Let us write $F(x)$ for an antiderivative of $f(x)$.

- The DE $\frac{\mathrm{d}u}{\mathrm{d}x} = u^2 f(x)$ from the first part is separable: $u^2 \mathrm{d}u = f(x)\mathrm{d}x$.
  After integration, we find $-\frac{1}{u} = F(x) + C$.
  Since $u = \frac{y}{x}$, this becomes $-\frac{x}{y} = F(x) + C$.
  The general solution of the initial DE therefore is $y = -\frac{x}{F(x) + C}$.

- The DE $x\frac{\mathrm{d}v}{\mathrm{d}x} = -v - f(x)$ from the second part is linear. We apply our recipe:

  (a) Rewrite the DE as $\frac{\mathrm{d}v}{\mathrm{d}x} + P(x)v = Q(x)$ with $P(x) = 1/x$ and $Q(x) = -f(x)/x$.

  (b) The integrating factor is $\exp\left(\int P(x)\mathrm{d}x\right) = e^{\ln x} = x$.

  Comment. We should make a mental note that we assumed that $x > 0$. In the next step, however, we see that the integrating factor works for all x.

  (c) Multiply the (rewritten) DE by the integrating factor x to get $\underbrace{x\frac{\mathrm{d}v}{\mathrm{d}x} + v}_{= \frac{\mathrm{d}}{\mathrm{d}x}[xv]} = -f(x)$.

  (d) Integrate both sides to get $xv = -F(x) + C$.

  Since $v = \frac{1}{y}$, we find $\frac{x}{y} = -F(x) + C$.
  The general solution of the initial DE therefore is $y = -\frac{x}{F(x) - C}$.

Comment. Note that our two approaches led to the same general solution (from the existence and uniqueness theorem, we can see that this must be the case). One of the formulas features $+C$ while the other features $-C$. However, that makes no difference because C is a free parameter (we could have given them different names if we preferred).

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Useful substitutions

The previous example illustrates that different substitutions can help to solve a given DE.

Choosing the right substitution is difficult in general. The following is a compilation of important cases that are easy to spot and for which the listed substitutions are guaranteed to succeed:

- $y' = F\left(\frac{y}{x}\right)$. This is called a **homogeneous equation**.

  Set $u = \frac{y}{x}$. Then $y = ux$ and $\frac{dy}{dx} = x\frac{du}{dx} + u$. We get $x\frac{du}{dx} + u = F(u)$. This DE is always separable.

  Caution. We will soon discuss homogeneous linear differential equations, where the label homogeneous means something different (though in both cases, there is a common underlying reason).

- $y' = F(ax + by)$

  Set $u = ax + by$. Then $y = \frac{1}{b}(u - ax)$ and $\frac{dy}{dx} = \frac{1}{b}\left(\frac{du}{dx} - a\right)$.

  The new DE is $\frac{1}{b}\left(\frac{du}{dx} - a\right) = F(u)$ or, simplified, $\frac{du}{dx} = a + bF(u)$. This DE is always separable.

- $y' = F(x)y + G(x)y^n$. This is called a **Bernoulli equation**.

  Set $u = y^{1-n}$. The resulting DE is always linear. We will consider this case next time.

- $F(y'', y', x) = 0$ (2nd order with "y missing")

  Set $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx}$. We get the first-order DE $F\left(\frac{du}{dx}, u, x\right) = 0$.

- $F(y'', y', y) = 0$ (2nd order with "x missing")

  Set $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = \frac{du}{dy} \cdot u$. We get the first-order DE $F\left(u\frac{du}{dy}, u, y\right) = 0$.

Example 34. (homework) Solve $\frac{dy}{dx} = (2x - 3y)^2 + \frac{2}{3}$, $y(1) = \frac{1}{3}$.

Solution. This is of the form $y' = F(2x - 3y)$ with $F(t) = t^2 + \frac{2}{3}$.

Therefore, as suggested by the above list, we substitute $u = 2x - 3y$.

Then $y = \frac{1}{3}(2x - u)$ and $\frac{dy}{dx} = \frac{1}{3}\left(2 - \frac{du}{dx}\right)$.

The new DE is $\frac{1}{3}\left(2 - \frac{du}{dx}\right) = u^2 + \frac{2}{3}$ or, simplified, $\frac{du}{dx} = -3u^2$.

This DE is separable: $u^{-2}du = -3dx$. After integration, $-\frac{1}{u} = -3x + C$.

We conclude that $u = \frac{1}{3x - C}$ and, hence, $y(x) = \frac{1}{3}(2x - u) = \frac{2}{3}x - \frac{1}{3}\frac{1}{3x - C}$.

Solving $y(1) = \frac{2}{3} - \frac{1}{3}\frac{1}{3 - C} = \frac{1}{3}$ for C leads to $C = 2$.

Hence, the unique solution of the IVP is $y(x) = \frac{2}{3}x - \frac{1}{3(3x - 2)}$.

Example 35. Consider $\frac{dy}{dx} = F(x)y + G(x)y^n$. This is called a **Bernoulli equation**. Substitute $u = y^{1-n}$ and show that the resulting linear DE.

Solution. If $u = y^{1-n}$ then $y = u^{1/(1-n)}$ and, thus, $\frac{dy}{dx} = \frac{1}{1-n} u^{n/(1-n)} \frac{du}{dx}$. $[\frac{1}{1-n} - 1 = \frac{n}{1-n}]$

The new DE is $\frac{1}{1-n} u^{n/(1-n)} \frac{du}{dx} = F(x)u^{1/(1-n)} + G(x)u^{n/(1-n)}$.

Dividing both sides by $u^{n/(1-n)}$, the DE simplifies to $\frac{1}{1-n}\frac{du}{dx} = F(x)u + G(x)$.

Comment. The original DE has the trivial solution $y = 0$. Do you see where we might lose that solution?

Example 36. (homework) Solve the IVP $\frac{dy}{dx} = 2y - 3xy^5$, $y(0) = 1$.

Solution. This is an example of a Bernoulli equation (with $n = 5$). We therefore substitute $u = y^{1-n} = y^{-4}$. Accordingly, $y = u^{-1/4}$ and, thus, $\frac{dy}{dx} = -\frac{1}{4} u^{-5/4} \frac{du}{dx}$.

The new DE is $-\frac{1}{4} u^{-5/4} \frac{du}{dx} = 2u^{-1/4} - 3xu^{-5/4}$, which simplifies to $\frac{du}{dx} = -8u + 12x$.

This is a linear first-order DE, which we solve according to our recipe:

(a) Rewrite the DE as $\frac{du}{dx} + P(x)u = Q(x)$ with $P(x) = 8$ and $Q(x) = 12x$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)dx\right) = e^{8x}$.

(c) Multiply the (rewritten) DE by $f(x) = e^{8x}$ to get

$$\underbrace{e^{8x}\frac{du}{dx} + 8e^{8x}u}_{= \frac{d}{dx}[e^{8x}u]} = 12xe^{8x}.$$

(d) Integrate both sides to get:

$$e^{8x}u = 12\int xe^{8x}dx = 12\left(\frac{1}{8}xe^{8x} - \frac{1}{8^2}e^{8x}\right) + C = \frac{3}{2}xe^{8x} - \frac{3}{16}e^{8x} + C$$

Here we used that $\int xe^{ax}dx = \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax}$. (Integration by parts!)

The general solution of the DE for u therefore is $u = \frac{3}{2}x - \frac{3}{16} + Ce^{-8x}$.

Correspondingly, the general solution of the initial DE is $y = u^{-1/4} = 1/\sqrt[4]{\frac{3}{2}x - \frac{3}{16} + Ce^{-8x}}$.

Using $y(0) = 1$, we find $1 = 1/\sqrt[4]{C - \frac{3}{16}}$ from which we obtain $C = 1 + \frac{3}{16} = \frac{19}{16}$.

The unique solution to the IVP therefore is $y = 1/\sqrt[4]{\frac{3}{2}x - \frac{3}{16} + \frac{19}{16}e^{-8x}}$.

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Example 37. Solve $(x-y)\frac{dy}{dx} = x+y$.

Solution. Divide the DE by x to get $(1-\frac{y}{x})\frac{dy}{dx} = 1+\frac{y}{x}$. This is a homogeneous equation!

We therefore substitute $u = \frac{y}{x}$. Then $y = ux$ and $\frac{dy}{dx} = x\frac{du}{dx} + u$.

The resulting DE is $(x-ux)\left(x\frac{du}{dx}+u\right) = x+ux$, which simplifies to $x(1-u)\frac{du}{dx} = 1+u^2$.

This DE is separable: $\frac{1-u}{1+u^2}du = \frac{1}{x}dx$

Integrating both sides, we find $\arctan(u) - \frac{1}{2}\ln(1+u^2) = \ln|x| + C$.

Setting $u = y/x$, we get the (general) implicit solution $\arctan(y/x) - \frac{1}{2}\ln(1+(y/x)^2) = \ln|x| + C$.

Comment. We used $\int \frac{1}{1+u^2}du = \arctan(u) + C$ and $\int \frac{x}{1+x^2}dx = \frac{1}{2}\ln(1+x^2) + C$ when integrating.

See Example 31 where we reviewed these integrals.

Solving simple 2nd order DEs

We have the following two useful substitutions for certain simple DEs of order 2:

- $F(y'', y', x) = 0$ (2nd order with "y missing")

  Set $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx}$. We get the first-order DE $F\left(\frac{du}{dx}, u, x\right) = 0$.

- $F(y'', y', y) = 0$ (2nd order with "x missing")

  Set $u = y' = \frac{dy}{dx}$. Then $y'' = \frac{du}{dx} = \frac{du}{dy}\cdot\frac{dy}{dx} = \frac{du}{dy}\cdot u$. We get the first-order DE $F\left(u\frac{du}{dy}, u, y\right) = 0$.

Example 38. Solve $y'' = x - y'$.

Solution. We substitute $u = y'$, which results in the first-order DE $u' = x - u$.

This DE is linear and, using our recipe (see below for the details), we can solve it to find $u = x - 1 + Ce^{-x}$.

Since $y' = u$, we conclude that the general solution is $y = \int (x-1+Ce^{-x})dx = \frac{1}{2}x^2 - x - Ce^{-x} + D$.

Important comment. This is a DE of order 2. Hence, as expected, the general solution has two free parameter.

Solving the linear DE. To solve $u' = x - u$ (also see Example 28, where we had solved this DE before), we

(a) rewrite the DE as $\frac{du}{dx} + P(x)u = Q(x)$ with $P(x) = 1$ and $Q(x) = x$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)dx\right) = e^x$.

(c) Multiply the (rewritten) DE by $f(x) = e^x$ to get $\underbrace{e^x\frac{du}{dx} + e^x u}_{=\frac{d}{dx}[e^x u]} = xe^x$.

(d) Integrate both sides to get (using integration by parts): $e^x u = \int xe^x dx = xe^x - e^x + C$

Hence, the general solution of the DE for u is $u = x - 1 + Ce^{-x}$, which is what we used above.

Example 39. (homework) Solve the IVP $y'' = x - y'$, $y(0) = 1$, $y'(0) = 2$.

Solution. As in the previous example, we find that the general solution to the DE is $y(x) = \frac{1}{2}x^2 - x - Ce^{-x} + D$.

Using $y'(x) = x - 1 + Ce^{-x}$ and $y'(0) = 2$, we find that $2 = -1 + C$. Hence, $C = 3$.

Then, using $y(x) = \frac{1}{2}x^2 - x - 3e^{-x} + D$ and $y(0) = 1$, we find $1 = -3 + D$. Hence, $D = 4$.

In conclusion, the unique solution to the IVP is $y(x) = \frac{1}{2}x^2 - x - 3e^{-x} + 4$.

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Example 40. (extra) Find the general solution to $y'' = 2yy'$.

Solution. We substitute $u = y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. Then $y'' = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot u$.

Therefore, our DE turns into $u\frac{\mathrm{d}u}{\mathrm{d}y} = 2yu$.

Dividing by u, we get $\frac{\mathrm{d}u}{\mathrm{d}y} = 2y$. [Note that we lose the solution $u = 0$, which gives the singular solution $y = C$.]

Hence, $u = y^2 + C$. It remains to solve $y' = y^2 + C$. This is a separable DE.

$\frac{1}{C + y^2}\mathrm{d}y = \mathrm{d}x$. Let us restrict to $C = D^2 \geqslant 0$ here. (This means we will only find "half" of the solutions.)

$\int \frac{1}{D^2 + y^2}\mathrm{d}y = \frac{1}{D^2}\int \frac{1}{1 + (y/D)^2}\mathrm{d}y = \frac{1}{D}\arctan(y/D) = x + A$.

Solving for y, we find $y = D\tan(Dx + AD) = D\tan(Dx + B)$. [$B = AD$]

Applications of DEs & Modeling

The exponential model of population growth

If $P(t)$ is the size of a population (eg. of bacteria) at time t, then the rate of change $\frac{\mathrm{d}P}{\mathrm{d}t}$ might, from biological considerations, be (nearly) proportional to $P(t)$.

Comment. "Population" might sound more specific than it is. It could also refer to rather different populations such as amounts of money (finance) or amounts of radioactive material (physics).

For instance, thinking about an amount $P(t)$ of money in a bank account at time t, we would also expect $\frac{\mathrm{d}P}{\mathrm{d}t}$ (the money per time that we gain from receiving interest) to be proportional to $P(t)$.

The corresponding **mathematical model** is described by the DE $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ where k is the constant of proportionality.

Example 41. Determine all solutions to the DE $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$.

Solution. We easily guess and then verify that $P(t) = Ce^{kt}$ is a solution. (Alternatively, we can find this solution via separation of variables or because this is a linear DE. Do it both ways!)

Moreover, it follows from the existence and uniqueness theorem that there cannot be further solutions. (Alternatively, we can conclude this from our solving process (separation of variables or our approach to linear DEs only lose solutions when we divide by zero and we can consider those cases separately)).

Mathematics therefore tells us that the (only) solutions to this DE are given by $P(t) = Ce^{kt}$ where C is some constant.

Hence, populations satisfying the assumption from biology necessarily exhibit exponential growth.

Example 42. Let $P(t)$ describe the size of a population at time t. Suppose $P(0) = 100$ and $P(1) = 300$. Under the exponential model of population growth, find $P(t)$.

Solution. $P(t)$ solves the DE $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ and therefore is of the form $P(t) = Ce^{kt}$.

We now use the two data points to determine both C and k.

$Ce^{k \cdot 0} = C = 100$ and $Ce^k = 100e^k = 300$. Hence $k = \ln(3)$ and $P(t) = 100e^{\ln(3)t} = 100 \cdot 3^t$.

Main problem of modeling: a model has to be detailed enough to resemble the real world, yet simple enough to allow for mathematical analysis.

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The logistic model of population growth

If the population is constrained by resources, then $\frac{dP}{dt} = kP$ is not a good model. A model to take that into account is $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$. This is the **logistic equation**.

M is called the carrying capacity:

- Note that if $P \ll M$ then $1 - \frac{P}{M} \approx 1$ and we are back to the simpler exponential model. This means that the population P will grow (nearly) exponentially if P is much less than the carrying capacity M.
- On the other hand, if $P > M$ then $1 - \frac{P}{M} < 0$ so that (assuming $k > 0$) $\frac{dP}{dt} < 0$, which means that the population P is shrinking if it exceeds the carrying capacity M.

Comment. If $P(t)$ is the size of a population, then P'/P can be interpreted as its *per capita growth rate*. Note that in the exponential model we have that $P'/P = k$ is constant.
On the other hand, in the logistic model we have that $P'/P = k(1 - P/M)$ is a linear function.

Example 43. Solve the logistic equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$.

Solution. This is a separable DE: $\frac{1}{P(1 - \frac{P}{M})} dP = k dt$.

To integrate the left-hand side, we use partial fractions: $\frac{1}{P(1 - \frac{P}{M})} = \frac{1}{P} + \frac{1/M}{1 - \frac{P}{M}} = \frac{1}{P} - \frac{1}{P - M}$.

After integrating, we obtain $\ln|P| - \ln|P - M| = kt + A$.

Equivalently, $\ln\left|\frac{P}{P - M}\right| = kt + A$ so that $\frac{P}{P - M} = \pm e^{kt + A} = Be^{kt}$ where $B = \pm e^A$.

Solving for P, we conclude that the general solution is

$$P(t) = \frac{BMe^{kt}}{Be^{kt} - 1} = \frac{M}{1 + Ce^{-kt}},$$

where replaced the free parameter B with $C = -1/B$.

Initial population. Note that the initial population is $P(0) = \frac{M}{1 + C}$. Equivalently, $C = \frac{M}{P(0)} - 1$ which expresses the free parameter C in terms of the initial population.

Comment. Note that $B = \pm e^A$ can be any real number except 0. However, we can easily check that $B = 0$ also gives us a solution to the DE (namely, the trivial solution $P = 0$). This solution was "lost" when we divided by P to separate variables.

Exercise. Note that the logistic equation is a Bernoulli equation. As an alternative to separation of variables, we can therefore solve it by transforming it to a linear DE.

Review of partial fractions. Recall that partial fractions tells us that fractions like $\frac{p(x)}{(x - r_1)(x - r_2)\cdots}$ (with the numerator of smaller degree than the denominator; and with the r_j distinct) can be written as a sum of terms of the form $\frac{A_j}{x - r_j}$ for suitable constants A_j.

In our case, this tells us that $\frac{1}{P(1 - P/M)} = \frac{A}{P} + \frac{B}{1 - P/M}$ for certain constants A and B.

Multiply both sides by P and set $P = 0$ to find $A = 1$.

Multiply both sides by $1 - P/M$ and set $P = M$ to find $B = 1/M$. This is what we used above.

The **logistic equation** with growth rate k and carrying capacity M is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right).$$

The general solution is $P(t) = \frac{M}{1 + Ce^{-kt}}$ where $C = \frac{M}{P(0)} - 1$.

Example 44. Let $P(t)$ describe the size of a population at time t. Under the logistic model of population growth, what is $\lim_{t\to\infty} P(t)$?

Solution.

- If $k>0$, then $e^{-kt}\to 0$ and it follows from $P(t)=\frac{M}{1+Ce^{-kt}}$ that $\lim_{t\to\infty} P(t)=M$.

  In other words, the population will approach the carrying capacity in the long run.

- If $k=0$, then we simply have $P(t)=\frac{M}{1+C}$. In other words, the population remains constant.

  This is a corner case because the DE becomes $\frac{\mathrm{d}P}{\mathrm{d}t}=0$.

- If $k<0$, then $e^{-kt}\to\infty$ and it follows that $\lim_{t\to\infty} P(t)=0$.

  In other words, the population will approach extinction in the long run.

Example 45. (homework) A rising population is modeled by the equation $\frac{\mathrm{d}P}{\mathrm{d}t}=400P-2P^2$.

(a) When the population size stabilizes in the long term, how big will the population be?

(b) Under which condition will the population size shrink?

(c) What is the population size when it is growing the fastest?

(d) If $P(0)=10$, what is $P(t)$?

Solution.

(a) Once the population reaches a stable level in the long term, we have $\frac{\mathrm{d}P}{\mathrm{d}t}=0$ (no change in population size).
Hence, $0=400P-2P^2=2P(200-P)$ which implies that $P=0$ or $P=200$. Since the population is rising, it will approach 200 in the long term.
Alternatively. Our DE matches the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t}=kP\left(1-\frac{P}{M}\right)$ with $k=400$ and $M=200$.

(b) The population size will shrink if $\frac{\mathrm{d}P}{\mathrm{d}t}<0$.
The DE tells us that is the case if and only if $400P-2P^2<0$ or, equivalently, if $P>\frac{400}{2}=200$.
Comment. In the logistic model, the population shrinks if it exceeds the carrying capacity.

(c) This is asking when $\frac{\mathrm{d}P}{\mathrm{d}t}$ (the population growth) is maximal.
The DE is telling us that this growth is $f(P)=400P-2P^2$. This a parabola that opens to the bottom. From Calculus, we know that it has a global maximum when $f'(P)=0$.
$f'(P)=400-4P=0$ leads to $P=100$.
Thus, the population is growing the fastest when its size is 100.
Comment. In the logistic model, the population is growing fastest when it is half the carrying capacity.

(d) We know that the general solution of the logistic equation is $P(t)=\frac{M}{1+Ce^{-kt}}=\frac{200}{1+Ce^{-400t}}$.
Using $P(0)=10$, we find that $C=\frac{200}{10}-1=19$.
Thus $P(t)=\frac{200}{1+19e^{-400t}}$.

Example 46. A scientist is claiming that a certain population $P(t)$ follows the logistic model of population growth perfectly. How many data points do you need to begin to verify that claim?

Solution. The general solution $P(t)=\frac{M}{1+Ce^{-kt}}$ to the logistic equation has 3 parameters.

Hence, we need 3 data points just to solve for their values.

Once we have 4 or more data points, we are able to test whether $P(t)$ conforms to the logistic model.

Important comment. Complicated models tend to have many degrees of freedom, which makes it easier to fit them to real world data (even if the model is not actually particularly appropriate). We therefore need to be cognizant about how much evidence is needed to decide that a given model is appropriate for the data.

Further population models

Let $P(t)$ be the size of the population that we wish to model at time t.

Denote with $\beta(t)$ and $\delta(t)$ the birth and death rate at time t, measured in number of births or deaths per unit of population per unit of time.

In the time interval $[t, t+\Delta t]$, we have that

$$\Delta P \approx \beta(t)P(t)\Delta t - \delta(t)P(t)\Delta t.$$

Comment. The reason that this is not an exact equation is that the rates $\beta(t)$ and $\delta(t)$ are allowed to change with t. In the above, we used these rates at time t for all times in $[t, t+\Delta t]$. This is a good approximation if Δt is small.

Divide both sides by Δt and let $\Delta t \to 0$ to obtain the general differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta(t) - \delta(t))P.$$

Given certain scenarios, we now make corresponding reasonable choices for $\beta(t)$ and $\delta(t)$.

- **(basic)** If the rates $\beta(t)$ and $\delta(t)$ are constant over time, the DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta - \delta)P$.
  This is the exponential model of population growth.

- **(limited supply)** If supply is limited, the birth rate will decrease as P increases. The simplest such relationship would be a linear dependence, which would take the form $\beta(t) = \beta_0 - \beta_1 P$.

  On the other hand, we still assume that $\delta(t)$ is constant. (However, depending on circumstances, it could also be reasonable to assume that $\delta(t)$ increases as P increases.)

  With these assumptions, the corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta_0 - \beta_1 P - \delta)P$.

  This is the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(1 - P/M)$ with $k = \beta_0 - \delta$ and $\frac{k}{M} = \beta_1$.

- **(rare isolated species)** If the population consists of rare and isolated specimen which rely on chance encounters to reproduce, then it is reasonable to assume that the birth rate $\beta(t)$ is proportional to $P(t)$ (larger $P(t)$ means more possibilities for chance encounters). Once more, we assume that $\delta(t)$ constant.

  With these assumptions, the corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (kP - \delta)P$.

  This is, again, the logistic equation.

- **(rare isolated species with very long life)** As before, for a rare isolated population, it is reasonable to assume that $\beta(t)$ is proportional to $P(t)$. If, in addition, our specimen have very long life, then we would assume that $\delta(t) = 0$.

  The corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP^2$. Solutions are $P(t) = \frac{1}{C - kt}$ where $P(0) = 1/C$. (Do it!)

  Comment. Note that $P(t) \to \infty$ as $t \to C/k$. This explosion (which implies population growth beyond exponential growth) emphasizes that we can only use the DE while our initial assumptions are satisfied. Here, the DE is no longer appropriate when our species is no longer rare because $P(t)$ is too large.

- **(spread of contagious incurable virus)** Let $P(t)$ count the number of infected population units among a (constant) total of N. Since the virus is incurable, we have $\delta(t) = 0$. On the other hand, it is reasonable to assume that $\beta(t)$ is proportional to $N - P$ (the number of people that can still be infected).

  The resulting DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(N - P)$. Once again, this is the logistic equation.

- **(harvesting)** Suppose that h population units are harvested each unit of time.

  Then the DE becomes $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta(t) - \delta(t))P - h$.

  For instance. $\frac{\mathrm{d}P}{\mathrm{d}t} = kP - h$ has the solution $P(t) = Ce^{kt} + h/k$. In that case, we get exponential growth if $C > 0$. Note that $P(0) = C + h/k$. In terms of the initial population $P(0)$, we therefore get exponential growth if $P(0) > h/k$.

Review. The **logistic equation** is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$.

Here, k is the growth rate and M is the carrying capacity.

The general solution is $P(t) = \frac{M}{1 + Ce^{-kt}}$ where $C = \frac{M}{P(0)} - 1$.

Example 47. In a city with a fixed population N, the time rate of change of the number P of people who have heard a certain rumor is proportional to the product of P and $N - P$. Suppose initially 10% have heard the rumor and after a week this number has grown to 20%. What percentage will this number reach after one more week?

Solution. $\frac{\mathrm{d}P}{\mathrm{d}t} = \gamma P(N - P)$. $P(0) = 0.1N$ and $P(1) = 0.2N$. We need $P(2)$.

Note that this is a logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)$ with $k = \gamma N$ and carrying capacity N.

It therefore has the general solution $P(t) = \frac{N}{1 + Ce^{-kt}}$.

Using $P(0) = \frac{N}{1 + C} = 0.1N$, we find that $C = 9$.

Using $P(1) = \frac{N}{1 + 9e^{-k}} = 0.2N$, we further find that $e^{-k} = \frac{4}{9}$.

We could solve for k but note that it is more pleasing to use $e^{-kt} = (e^{-k})^t = \left(\frac{4}{9}\right)^t$ in our formula for $P(t)$.

We conclude that $P(t) = \frac{N}{1 + 9\left(\frac{4}{9}\right)^t}$.

In particular, $P(2) = \frac{N}{1 + 9 \cdot \frac{16}{81}} = \frac{9}{25}N$ which is 36%.

Mixing problems

Example 48. A tank contains 20gal of pure water. It is filled with brine (containing 5lb/gal salt) at a rate of 3gal/min. At the same time, well-mixed solution flows out at a rate of 2gal/min. How much salt is in the tank after t minutes?

Solution. Let $x(t)$ denote the amount of salt (in lb) in the tank after time t (in min).

At time t, the concentration of salt (in lb/gal) in the tank is $\frac{x(t)}{V(t)}$ where $V(t) = 20 + (3 - 2)t = 20 + t$ is the volume (in gal) in the tank.

In the time interval $[t, t + \Delta t]$: $\Delta x \approx 3 \cdot 5 \cdot \Delta t - 2 \cdot \frac{x(t)}{V(t)} \cdot \Delta t$.

Hence, $x(t)$ solves the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = 15 - 2 \cdot \frac{x}{20 + t}$ with $x(0) = 0$. Since this is a linear DE, we can solve it as follows:

- We write it in the form $\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{2}{20 + t}x = 15$.
- The integrating factor is $f(t) = \exp\left(\int \frac{2}{20 + t}\mathrm{d}t\right) = (20 + t)^2$.
- Multiply the (rewritten) DE by $f(t) = (20 + t)^2$ to get $\underbrace{(20 + t)^2\frac{\mathrm{d}x}{\mathrm{d}t} + 2(20 + t)x}_{= \frac{\mathrm{d}}{\mathrm{d}t}[(20+t)^2 x]} = 15(20 + t)^2$.
- Integrate both sides to get $(20 + t)^2 x = 15\int (20 + t)^2 \mathrm{d}t = 5(20 + t)^3 + C$.

Hence the general solution to the DE is $x(t) = 5(20 + t) + \frac{C}{(20 + t)^2}$. Using $x(0) = 0$, we find $C = -5 \cdot 20^3$.

We conclude that, after t minutes, the tank contains $x(t) = 5(20 + t) - \frac{5 \cdot 20^3}{(20 + t)^2}$ pounds of salt.

Comment. As a consequence, $x(t) \approx 5(20 + t) = 5V(t)$ for large t. Why does that make perfect sense?!

Acceleration–velocity models

To model a falling object, we let $y(t)$ be its height at time t.

Then physics has names for $y'(t)$ and $y''(t)$: these are the **velocity** and the **acceleration**.

Physics tells us that objects fall due to gravity (and that it makes already falling objects fall faster; in other words, gravity accelerates falling objects). Physicists have measured that, on earth, the the graviational acceleration is $g \approx 9.81 \mathrm{m/s}^2$.

If we only take earth's gravitation into account, then the fall is therefore modelled by

$$y''(t) = -g.$$

Example 49. A ball is dropped from a 100m tall building. How long until it reaches the ground? What is the speed when it hits the ground?

Solution. Let $y(t)$ be the height (in meters) at which the ball is at time t (in seconds).

As above, physics tells us that an object falling due to gravity (and ignoring everything else) satisfies the DE $y'' = -g$ where $g \approx 9.81$. We further know the initial values $y(0) = 100$, $y'(0) = 0$.

Substituting $v = y'$ in the DE, we get $v' = -g$. This DE is solved by $v(t) = -gt + C$.

Hence, $y(t) = \int v(t) \mathrm{d}t = -\frac{1}{2} g t^2 + Ct + D$.

The initial conditions $y(0) = 100$, $y'(0) = 0$ tell us that $D = 100$ and $C = 0$.

Thus $y(t) = -\frac{1}{2} g t^2 + 100$.

The ball reaches the ground when $y(t) = -\frac{1}{2} g t^2 + 100 = 0$, that is after $t = \sqrt{200/g} \approx 4.52$ seconds.

The speed then is $|y'(4.5)| \approx 44.3 \, \mathrm{m}/s$.

For many applications, one needs to take air resistance into account.

This is actually less well understood than one might think, and the physics quickly becomes rather complicated. Typically, air resistance is somewhere in between the following two cases:

- Under certain assumptions, physics suggests that air resistance is proportional to the square of the velocity.

  Comment. A simplistic way to think about this is to imagine the falling object to bump into (air) particles; if the object falls twice as fast, then the momentum of the particles it bumps into is twice as large and it bumps into twice as many of them.

- In other cases such as "relatively slowly" falling objects, one might empirically observe that air resistance is proportional to the velocity itself.

  Comment. One might imagine that, at slow speed, the falling object doesn't exactly bump into particles but instead just gently pushes them aside; so that at twice the speed it only needs to gently push twice as often.

Example 50. When modeling the (slow) fall of a parachute, physics suggests that the air resistance is roughly proportional to velocity. If $y(t)$ is the parachute's height at time t, then the corresponding DE is $y'' = -g - \rho y'$ where $\rho > 0$ is a constant.

Comment. Note that $-\rho y' > 0$ because $y' < 0$. Thus, as intended, air resistance is acting in the opposite direction as gravity and slowing down the fall.

Determine the general solution of the DE.

Solution. Substituting $v = y'$, the DE becomes $v' + \rho v = -g$.

This is a linear DE. To solve it, we determine that the integrating factor is $\exp(\int \rho \mathrm{d}t) = e^{\rho t}$.

Multiplying the DE with that factor and integrating, we obtain $e^{\rho t} v = \int -g e^{\rho t} \mathrm{d}t = -\frac{g}{\rho} e^{\rho t} + C$.

Hence, $v(t) = -\frac{g}{\rho} + C e^{-\rho t}$.

Correspondingly, the general solution of the DE is $y(t) = \int v(t) \mathrm{d}t = -\frac{g}{\rho} t - \frac{C}{\rho} e^{-\rho t} + D$.

Comment. Note that $\lim_{t \to \infty} v(t) = -\frac{g}{\rho}$. In other words, the **terminal velocity** is $v_\infty = -\frac{g}{\rho}$.

This is an interesting mathematical consequence of the DE. (And important for the idea behind a parachute!)

Note that, if we know that there is a terminal speed, then we can actually determine its value v_∞ from the DE without solving it by setting $v' = 0$ (because, once the terminal speed is reached, the velocity does not change anymore) in $v' + \rho v = -g$. This gives us $\rho v_\infty = -g$ and, hence, $v_\infty = -g / \rho$ as above.

Let us have another look at Example 6. Note that the DE is a second-order linear differential equation with constant coefficients. Our upcoming goal will be to solve all such equations.

Example 51. Find the general solution to $y'' = y' + 6y$.

Solution. We look for solutions of the form e^{rx}.

Plugging e^{rx} into the DE, we get $r^2 e^{rx} = r e^{rx} + 6e^{rx}$ which simplifies to $r^2 - r - 6 = 0$.

This is called the **characteristic equation**. Its solutions are $r = -2, 3$ (the **characteristic roots**).

This means we found the two solutions $y_1 = e^{-2x}$, $y_2 = e^{3x}$.

The general solution to the DE is $C_1 e^{-2x} + C_2 e^{3x}$.

Comment. In the final step, we used an important principle that is true for linear (!) homogeneous DEs. Namely, if we have solutions $y_1, y_2, \ldots$ then any linear combination $C_1 y_1 + C_2 y_2 + \ldots$ is a solution as well. We will discuss this soon but, for now, check that $C_1 e^{-2x} + C_2 e^{3x}$ is indeed a solution by plugging it into the DE.

Spotlight on the exponential function

Example 52. Solve $y' = ky$ where k is a constant.

Solution. (experience) At this point, we can probably see that $y(x) = e^{kx}$ is a solution.

In fact, the general solution is $y(x) = Ce^{kx}$.

That there cannot be any further solutions follows from the existence and uniqueness theorem (see next example).

Solution. (separation of variables) Alternatively, we can solve the DE using separation of variables.

Express the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} = ky$, then write it as $\frac{1}{y}\mathrm{d}y = k\,\mathrm{d}x$ (note that we just lost the solution $y = 0$).

Integrating gives $\ln|y| = kx + D$, hence $|y| = e^{kx+D}$.

Since the RHS is never zero, $y = \pm e^{kx+D} = Ce^{kx}$ (with $C = \pm e^D$). Finally, note that $C = 0$ corresponds to the singular solution $y = 0$ that we lost. In summary, the general solution is Ce^{kx}.

Example 53. Consider the IVP $y' = ky$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = ky$. We compute that $\frac{\partial}{\partial y} f(x, y) = k$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y).

Hence, for any initial conditions, the IVP locally has a unique solution by the existence and uniqueness theorem.

Comment. As a consequence, there can be no other solutions to the DE $y' = ky$ than the ones of the form $y(x) = Ce^{kx}$. Why?! [Assume that $y(x)$ satisfies $y' = ky$ and let (a, b) any value on the graph of y. Then $y(x)$ solves the IVP $y' = ky$, $y(a) = b$; but so does Ce^{kx} with $C = b/e^{ka}$. The uniqueness implies that $y(x) = Ce^{kx}$.]

In particular, we have the following characterization of the exponential function:

e^x is the unique solution to the IVP $y' = y$, $y(0) = 1$.

Comment. Note that, for instance, $\frac{\mathrm{d}}{\mathrm{d}x} 2^x = \ln(2)\, 2^x$. (This follows from $2^x = e^{\ln(2^x)} = e^{x\ln(2)}$.)

Since $\ln = \log_e$, this means that we cannot avoid the natural base $e \approx 2.718$ even if we try to use another base.

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Excursion: Euler's identity

Theorem 54. (Euler's identity) $e^{ix} = \cos(x) + i\sin(x)$

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.
[Check that by computing the derivatives and verifying the initial condition! As we did in class.] □

On lots of T-shirts. In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Example 55. Where do trig identities like $\sin(2x) = 2\cos(x)\sin(x)$ or $\sin^2(x) = \frac{1-\cos(2x)}{2}$ (and infinitely many others!) come from?

Short answer: they all come from the simple exponential law $e^{x+y} = e^x e^y$.

Let us illustrate this in the simple case $(e^x)^2 = e^{2x}$. Observe that

$$\begin{aligned} e^{2ix} &= \cos(2x) + i\sin(2x) \\ e^{ix}e^{ix} &= [\cos(x) + i\sin(x)]^2 = \cos^2(x) - \sin^2(x) + 2i\cos(x)\sin(x). \end{aligned}$$

Comparing imaginary parts (the "stuff with an i"), we conclude that $\sin(2x) = 2\cos(x)\sin(x)$.
Likewise, comparing real parts, we read off $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(Use $\cos^2(x) + \sin^2(x) = 1$ to derive $\sin^2(x) = \frac{1-\cos(2x)}{2}$ from the last equation.)

Challenge. Can you find a triple-angle trig identity for $\cos(3x)$ and $\sin(3x)$ using $(e^x)^3 = e^{3x}$?
Or, use $e^{i(x+y)} = e^{ix}e^{iy}$ to derive $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x+y) = \ldots$

Realize that the complex number $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ corresponds to the point $(\cos(\theta), \sin(\theta))$.
These are precisely the points on the unit circle!

Recall that a point (x, y) can be represented using **polar coordinates** (r, θ), where r is the distance to the origin and θ is the angle with the x-axis.

Then, $x = r\cos\theta$ and $y = r\sin\theta$.

Every complex number z can be written in **polar form** as $z = re^{i\theta}$, with $r = |z|$.

Why? By comparing with the usual polar coordinates ($x = r\cos\theta$ and $y = r\sin\theta$), we can write

$$z = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}.$$

In the final step, we used Euler's identity.

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Linear DEs of higher order

The most general linear first-order DE is of the form $A(x)y' + B(x)y + C(x) = 0$. Any such DE can be rewritten in the form $y' + P(x)y = Q(x)$ by dividing by $A(x)$ and rearranging.

We have learned how to solve all of these using an integrating factor.

Likewise, any **linear DE** of order n can be written in the form

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = Q(x).$$

The corresponding **homogeneous linear DE** is the DE

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = 0,$$

and it plays an important role in solving the original linear DE.

Comment. The homogeneous equations $y' = F(y/x)$ that we encountered earlier are something rather different. It is a bit unfortunate that the word homogeneous is used for both.

Homogeneous linear DE have the important property that, if y_1 and y_2 are two solutions, then the linear combination $C_1\, y_1 + C_2\, y_2$ is a solution as well.

Example 56. Suppose that y_1 and y_2 solve $y'' + P_1(x)y' + P_0(x)y = 0$. Show that $7y_1 + 4y_2$ is another solution of the DE.

Solution. $(7y_1 + 4y_2)'' + P_1(x)(7y_1 + 4y_2)' + P_0(x)(7y_1 + 4y_2)$
$= 7\{y_1'' + P_1(x)y_1' + P_0(x)y_1\} + 4\{y_2'' + P_1(x)y_2' + P_0(x)y_2\} = 0 + 0$

In other words, $7y_1 + 4y_2$ is another solution of the DE.

Comment. Of course, there is nothing special about the coefficients 7 and 4. The same argument shows that any linear combination $C_1\, y_1 + C_2\, y_2$ is a solution as well.

Important comment. Make sure that you see that it is crucial that the DE is linear and that it is homogeneous! (What happens if the DE is linear but not homogeneous?)

The upshot is that this observation reduces the task of finding the general solution of a homogeneous linear DE to the task of finding n (sufficiently) different solutions.

(general solution of a homogeneous linear DE) For any homogeneous linear DE of order n, there are n solutions $y_1, y_2, \ldots, y_n$ such that the general solution is

$$y(x) = C_1 y_1(x) + \ldots + C_n\, y_n(x).$$

Comment. As we observed in the first-order case, if I is an interval on which all the $P_j(x)$ as well as $P(x)$ are continuous, then for any $a \in I$ the IVP with $y(a) = b_0$, $y'(a) = b_1$, ..., $y^{(n-1)}(a) = b_{n-1}$ always has a unique solution (which is defined on all of I).

Example 57. (extra) The DE $x^2 y'' + 2x y' - 6y = 0$ has solutions $y_1 = x^2$, $y_2 = x^{-3}$. Solve the IVP with $y(2) = 10$, $y'(2) = 15$.

Solution. Note that this is a homogeneous linear DE of order 2.

Hence, given the two solutions, we conclude that the general solution is $y(x) = Ax^2 + Bx^{-3}$.

Using $y'(x) = 2Ax - 3Bx^{-4}$, the two initial conditions allow us to solve for A and B:

Solving $y(2) = 4A + B/8 = 10$ and $y'(2) = 4A - 3/16B = 15$ for A and B results in $A = 3$, $B = -16$.

So the unique solution to the IVP is $y(x) = 3x^2 - 16/x^3$.

Homogeneous linear DEs with constant coefficients

Let us start with another example like Examples 6 and 51. This time we also approach this computation using an operator approach that explains further what is going on (and that will be particularly useful when we discuss inhomogeneous equations).

An **operator** takes a function as input and returns a function as output. That is exactly what the derivative does.

In the sequel, we write $D = \frac{\mathrm{d}}{\mathrm{d}x}$ for the derivative operator.

For instance. We write $y' = \frac{\mathrm{d}}{\mathrm{d}x} y = D y$ as well as $y'' = \frac{\mathrm{d}^2}{\mathrm{d}x^2} y = D^2 y$.

Example 58. Find the general solution to $y'' - y' - 2y = 0$.

Solution. (our earlier approach) Let us look for solutions of the form e^{rx}.
Plugging e^{rx} into the DE, we get $r^2 e^{rx} - r e^{rx} - 2e^{rx} = 0$.
Equivalently, $r^2 - r - 2 = 0$. This is the characteristic equation. Its solutions are $r = 2, -1$.
This means we found the two solutions $y_1 = e^{2x}$, $y_2 = e^{-x}$.
Since this a homogeneous linear DE, the general solution is $y = C_1 e^{2x} + C_2 e^{-x}$.

Solution. (operator approach) $y'' - y' - 2y = 0$ is equivalent to $(D^2 - D - 2)y = 0$.
Note that $D^2 - D - 2 = (D - 2)(D + 1)$ is the **characteristic polynomial**.
Observe that we get solutions to $(D - 2)(D + 1)y = 0$ from $(D - 2)y = 0$ and $(D + 1)y = 0$.
$(D - 2)y = 0$ is solved by $y_1 = e^{2x}$, and $(D + 1)y = 0$ is solved by $y_2 = e^{-x}$; as in the previous solution.
Again, we conclude that the general solution is $y = C_1 e^{2x} + C_2 e^{-x}$.

Set $D = \frac{\mathrm{d}}{\mathrm{d}x}$. Every **homogeneous linear DE with constant coefficients** can be written as $p(D)y = 0$, where $p(D)$ is a polynomial in D, called the **characteristic polynomial**.

For instance. $y'' - y' - 2y = 0$ is equivalent to $Ly = 0$ with $L = D^2 - D - 2$.

Example 59. Solve $y'' - y' - 2y = 0$ with initial conditions $y(0) = 4$, $y'(0) = 5$.

Solution. From the previous example, we know that the general solution is $y(x) = C_1 e^{2x} + C_2 e^{-x}$.
Using $y'(x) = 2C_1 e^{2x} - C_2 e^{-x}$, the initial conditions result in the two equations $C_1 + C_2 = 4$, $2C_1 - C_2 = 5$.
Solving these we find $C_1 = 3$, $C_2 = 1$.
Hence the unique solution to the IVP is $y(x) = 3e^{2x} + e^{-x}$.

Review. Every homogeneous linear DE with constant coefficients can be written as $p(D)y=0$, where $D=\frac{\mathrm{d}}{\mathrm{d}x}$ and $p(D)$ is the characteristic polynomial polynomial.

Example 60. Determine the general solution of $y'''+7y''+14y'+8y=0$.

Solution. This DE is of the form $p(D)\,y=0$ with characteristic polynomial $p(D)=D^3+7D^2+14D+8$.

The characteristic polynomial factors as $p(D)=(D+1)(D+2)(D+4)$.

Hence, we found the solutions $y_1=e^{-x}$, $y_2=e^{-2x}$, $y_3=e^{-4x}$. Those are enough (independent!) solutions for a third-order DE. The general solution therefore is $y(x)=C_1\,e^{-x}+C_2\,e^{-2x}+C_3\,e^{-4x}$.

Example 61. Solve $y''=4y$ with initial conditions $y(0)=-1$, $y'(0)=10$.

Solution. This DE is of the form $p(D)\,y=0$ with characteristic polynomial $p(D)=D^2-4$.

The characteristic polynomial factors as $p(D)=(D-2)(D+2)$.

Hence, we found the solutions $y_1=e^{2x}$, $y_2=e^{-2x}$. Those are enough (independent!) solutions for a second-order DE. The general solution therefore is $y(x)=C_1\,e^{2x}+C_2\,e^{-2x}$.

Using $y'(x)=2C_1e^{2x}-2C_2e^{-2x}$, the initial conditions result in the equations $C_1+C_2=-1$, $2C_1-2C_2=10$.

Solving these we find $C_1=2$, $C_2=-3$.

Hence the unique solution to the IVP is $y(x)=2e^{2x}-3e^{-2x}$.

Comment (for those who have taken Linear Algebra). In matrix-vector notation, the two equations can be written and solved as

$$\begin{bmatrix}1 & 1\\ 2 & -2\end{bmatrix}\begin{bmatrix}C_1\\ C_2\end{bmatrix}=\begin{bmatrix}-1\\ 10\end{bmatrix} \quad\rightsquigarrow\quad \begin{bmatrix}C_1\\ C_2\end{bmatrix}=\begin{bmatrix}1 & 1\\ 2 & -2\end{bmatrix}^{-1}\begin{bmatrix}-1\\ 10\end{bmatrix}=-\frac{1}{4}\begin{bmatrix}-2 & -1\\ -2 & 1\end{bmatrix}\begin{bmatrix}-1\\ 10\end{bmatrix}=\begin{bmatrix}2\\ -3\end{bmatrix},$$

where we used the general formula for the inverse of a 2×2 matrix.

The system of equations $\begin{bmatrix}1 & 1\\ 2 & -2\end{bmatrix}\begin{bmatrix}C_1\\ C_2\end{bmatrix}=\begin{bmatrix}-1\\ 10\end{bmatrix}$ is an inhomogeneous system of linear equations.

The corresponding homogeneous system of linear equations is $\begin{bmatrix}1 & 1\\ 2 & -2\end{bmatrix}\begin{bmatrix}C_1\\ C_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$.

Example 62. (extra) Determine the general solution of $y'''-y''-4y'+4y=0$.

Solution. This DE is of the form $p(D)\,y=0$ with characteristic polynomial $p(D)=D^3-D^2-4D+4$.

The characteristic polynomial factors as $p(D)=(D-1)(D-2)(D+2)$.

Hence, we found the solutions $y_1=e^{x}$, $y_2=e^{2x}$, $y_3=e^{-2x}$. Those are enough (independent!) solutions for a third-order DE. The general solution therefore is $y(x)=C_1\,e^{x}+C_2\,e^{2x}+C_3\,e^{-2x}$.

In this manner, we are able to solve any homogeneous linear DE of order n with constant coefficients provided that there are n different roots r (each giving rise to one solution e^{rx}).

One issue is that roots might be repeated. In that case, we are currently missing solutions. The following example suggests how to get our hands on the missing solutions.

Example 63. Determine the general solution of $y'''=0$.

Solution. We know from Calculus that the general solution is $y(x)=C_1+C_2\,x+C_3\,x^2$.

Solution. (looking ahead) The characteristic polynomial $p(D)=D^3$ has roots $0,0,0$. By Theorem 64 below, we have the solutions $y(x)=x^j\,e^{0x}=x^j$ for $j=0,1,2$, so that the general solution is $y(x)=C_1+C_2\,x+C_3\,x^2$.

Theorem 64. Consider the homogeneous linear DE with constant coefficients $p(D)y=0$.

- If r is a root of the characteristic polynomial and if k is its multiplicity, then k (independent) solutions of the DE are given by $x^j e^{rx}$ for $j=0,1,\ldots,k-1$.
- Combining these solutions for all roots, gives the general solution.

This is because the order of the DE equals the degree of $p(D)$, and a polynomial of degree n has (counting with multiplicity) exactly n (possibly **complex**) roots.

In the complex case. Likewise, if $r=a\pm bi$ are roots of the characteristic polynomial and if k is its multiplicity, then $2k$ (independent) solutions of the DE are given by $x^j e^{ax}\cos(bx)$ and $x^j e^{ax}\sin(bx)$ for $j=0,1,\ldots,k-1$. This case will be discussed next time.

Proof. Let r be a root of the characteristic polynomial of multiplicity k. Then $p(D)=q(D)\,(D-r)^k$.

We need to find k solutions to the simpler DE $(D-r)^k y=0$.

It is natural to look for solutions of the form $y=c(x)e^{rx}$.

[This idea is called **variation of constants** since we know that this is a solution if $c(x)$ is a constant.]

Note that $(D-r)[c(x)e^{rx}]=(c'(x)e^{rx}+c(x)re^{rx})-rc(x)e^{rx}=c'(x)e^{rx}$.

Repeating, we get $(D-r)^2[c(x)e^{rx}]=(D-r)[c'(x)e^{rx}]=c''(x)e^{rx}$ and, eventually, $(D-r)^k[c(x)e^{rx}]=c^{(k)}(x)e^{rx}$.

In particular, $(D-r)^k y=0$ is solved by $y=c(x)e^{rx}$ if and only if $c^{(k)}(x)=0$.

The DE $c^{(k)}(x)=0$ is clearly solved by x^j for $j=0,1,\ldots,k-1$, and it follows that $x^j e^{rx}$ solves the original DE. □

Example 65. Determine the general solution of $y'''-3y''+3y'-y=0$.

Solution. The characteristic polynomial $p(D)=D^3-3D^2+3D-1=(D-1)^3$ has roots $1,1,1$.

By Theorem 64, the general solution is $y(x)=(C_1+C_2x+C_3x^2)e^x$.

Example 66. Determine the general solution of $y'''-3y'+2y=0$.

Solution. The characteristic polynomial $p(D)=D^3-3D+2=(D-1)^2(D+2)$ has roots $1,1,-2$.

By Theorem 64, the general solution is $y(x)=(C_1+C_2x)e^x+C_2e^{-2x}$.

Example 67. (homework) Solve the IVP $y'''=4y''-4y'$ with $y(0)=4$, $y'(0)=0$, $y''(0)=-4$.

Solution. The characteristic polynomial $p(D)=D^3-4D^2+4D=D(D-2)^2$ has roots $0,2,2$.

By Theorem 64, the general solution is $y(x)=C_1+(C_2+C_3x)e^{2x}$.

Using $y'(x)=(2C_2+C_3+2C_3x)e^{2x}$ and $y''(x)=4(C_2+C_3+C_3x)e^{2x}$, the initial conditions result in the equations $C_1+C_2=4$, $2C_2+C_3=0$, $4C_2+4C_3=-4$.

Solving these (start with the last two equations) we find $C_1=3$, $C_2=1$, $C_3=-2$.

Hence the unique solution to the IVP is $y(x)=3+(1-2x)e^{2x}$.

Review. A homogeneous linear DE with constant coefficients is of the form $p(D)y=0$, where $p(D)$ is the characteristic polynomial polynomial. For each characteristic root r of multiplicity k, we get the k solutions $x^j e^{rx}$ for $j=0,1,\ldots,k-1$.

Example 68. Determine the general solution of $y^{(6)}=3y^{(5)}-4y'''$.

Solution. This DE is of the form $p(D)\,y=0$ with $p(D)=D^6-3D^5+4D^3=D^3(D-2)^2(D+1)$.
The characteristic roots are $2,2,0,0,0,-1$.
By Theorem 64, the general solution is $y(x)=(C_1+C_2x)e^{2x}+C_3+C_4x+C_5x^2+C_6e^{-x}$.

Example 69. Consider the function $y(x)=3xe^{-2x}+7$. Determine a homogeneous linear DE with constant coefficients of which $y(x)$ is a solution.

Solution. In order for $y(x)$ to be a solution of $p(D)y=0$, the characteristic roots must include $-2,-2,0$.
The simplest choice for $p(D)$ thus is $p(D)=(D+2)^2D=D^3+4D^2+4D$.
Accordingly, $y(x)$ is a solution of $y'''+4y''+4y'=0$.

Example 70. (homework) Consider the function $y(x)=3xe^{-2x}+7e^x$. Determine a homogeneous linear DE with constant coefficients of which $y(x)$ is a solution.

Solution. In order for $y(x)$ to be a solution of $p(D)y=0$, the characteristic roots must include $-2,-2,1$.
The simplest choice for $p(D)$ thus is $p(D)=(D+2)^2(D-1)=D^3+3D^2-4$.
Accordingly, $y(x)$ is a solution of $y'''+3y''-4y=0$.

Real form of complex solutions

Let's recall some basic facts about **complex numbers**:

- Every complex number can be written as $z=x+iy$ with real x,y.
- Here, the imaginary unit i is characterized by solving $x^2=-1$.

  Important observation. The same equation is solved by $-i$. This means that, algebraically, we cannot distinguish between $+i$ and $-i$.

- The **conjugate** of $z=x+iy$ is $\bar{z}=x-iy$.

  Important comment. Since we cannot algebraically distinguish between $\pm i$, we also cannot distinguish between z and $\bar{z}$. That's the reason why, in problems involving only real numbers, if a complex number $z=x+iy$ shows up, then its **conjugate** $\bar{z}=x-iy$ has to show up in the same manner. With that in mind, have another look at the examples below.

- The **real part** of $z=x+iy$ is x and we write $\mathrm{Re}(z)=x$.
  Likewise the **imaginary part** is $\mathrm{Im}(z)=y$.

  Observe that $\mathrm{Re}(z)=\frac{1}{2}(z+\bar{z})$ as well as $\mathrm{Im}(z)=\frac{1}{2i}(z-\bar{z})$.

- **Euler's identity** (see Theorem 54) states that $e^{ix}=\cos(x)+i\sin(x)$.

  It follows that $\cos(x)=\mathrm{Re}(e^{ix})=\frac{1}{2}(e^{ix}+e^{-ix})$ and $\sin(x)=\mathrm{Im}(e^{ix})=\frac{1}{2i}(e^{ix}-e^{-ix})$.

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Example 71. Determine the general solution of $y'' + y = 0$.

Solution. The characteristic polynomial is $D^2 + 1$ which has no roots over the reals.
Over the **complex numbers**, by definition, the roots are i and $-i$.
So the general solution is $y(x) = C_1 e^{ix} + C_2 e^{-ix}$.

Solution. On the other hand, we easily check that $y_1 = \cos(x)$ and $y_2 = \sin(x)$ are two solutions.
Hence, the general solution can also be written as $y(x) = D_1 \cos(x) + D_2 \sin(x)$.

Important comment. That we have these two different representations is a consequence of Euler's identity (see Theorem 54)

$$e^{ix} = \cos(x) + i \sin(x).$$

Likewise, $e^{-ix} = \cos(x) - i \sin(x)$. (This follows from replacing x by $-x$ in Euler's identity.)

On the other hand, $\cos(x) = \frac{1}{2}(e^{ix} + e^{-ix})$ and $\sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$.

[Recall that the first formula is an instance of $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$ and the second of $\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$.]

Example 72. Determine the general solution of $y'' - 4y' + 13y = 0$ using only real numbers.

Solution. The characteristic polynomial $p(D) = D^2 - 4D + 13$ has roots $2 + 3i, 2 - 3i$.

[We can use the quadratic formula to find these roots as $\frac{4 \pm \sqrt{4^2 - 4 \cdot 13}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$.]

Hence, the general solution in real form is $y(x) = C_1 e^{2x} \cos(3x) + C_2 e^{2x} \sin(3x)$.

Note. $e^{(2+3i)x} = e^{2x} e^{3ix} = e^{2x}(\cos(3x) + i \sin(3x))$

Example 73. (review) Solve the IVP $y''' = 8y'' - 16y'$ with $y(0) = 1$, $y'(0) = 4$, $y''(0) = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - 8D^2 + 16D = D(D-4)^2$ has roots $0, 4, 4$.

By Theorem 64, the general solution is $y(x) = C_1 + (C_2 + C_3 x)e^{4x}$.

Using $y'(x) = (4C_2 + C_3 + 4C_3 x)e^{4x}$ and $y''(x) = 4(4C_2 + 2C_3 + 4C_3 x)e^{4x}$, the initial conditions result in the equations $C_1 + C_2 = 1$, $4C_2 + C_3 = 4$, $16C_2 + 8C_3 = 0$.

Solving these (start with the last two equations) we find $C_1 = -1$, $C_2 = 2$, $C_3 = -4$.

Hence the unique solution to the IVP is $y(x) = -1 + (2 - 4x)e^{4x}$.

Important comment. Check (as we did in class) that $y(x)$ indeed solves the IVP.

Example 74. (review) Determine the general solution of $y''' - y'' - 5y' - 3y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - D^2 - 5D - 3 = (D-3)(D+1)^2$ has roots $3, -1, -1$.

Hence, the general solution is $y(x) = C_1 e^{3x} + (C_2 + C_3 x)e^{-x}$.

Example 75. (review) Find the general solution of $y^{(7)} + 8y^{(6)} + 42y^{(5)} + 104y^{(4)} + 169y''' = 0$.

Use the fact that $-2 + 3i$ is a repeated characteristic root.

Solution. The characteristic polynomial $p(D) = D^3(D^2 + 4D + 13)^2$ has roots $0, 0, 0, -2 \pm 3i, -2 \pm 3i$.

[Since $-2 + 3i$ is a root so must be $-2 - 3i$. Repeating them once, together with $0, 0, 0$ results in 7 roots.]

Hence, the general solution is $(A + Bx + Cx^2) + (D + Ex)\, e^{-2x}\cos(3x) + (F + Gx)e^{-2x}\sin(3x)$.

Inhomogeneous linear DEs

Review. A linear DE of order n is of the form

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = Q(x).$$

- In terms of $D = \frac{\mathrm{d}}{\mathrm{d}x}$, the DE becomes: $Ly = f(x)$ with $L = D^n + P_{n-1}(x)D^{n-1} + \ldots + P_1(x)D + P_0(x)$.
- L is called a **linear differential operator**.
  - $L(c_1 y_1 + c_1 y_2) = c_1 L y_1 + c_2 L y_2$ **(linearity)**
    Comment. If you are familiar with linear algebra, think of L replaced with a matrix A and y_1, y_2 replaced with vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$. In that case, the same linearity property holds.
- The inclusion of the $Q(x)$ term makes $Ly = Q(x)$ an **inhomogeneous** linear DE.
- $Ly = 0$ is the corresponding **homogeneous** DE.
  - If y_1 and y_2 are solutions to the homogeneous DE, then so is any linear combination $C_1 y_1 + C_2 y_2$.
  - **(general solution of a homogeneous linear DE)** For any homogeneous linear DE of order n, there are n solutions $y_1, y_2, \ldots, y_n$ such that the general solution is $y(x) = C_1 y_1(x) + \ldots + C_n y_n(x)$.

(general solution of an inhomogeneous linear DE) The general solution of any inhomogeneous linear DE of order n is of the form

$$y(x) = y_p(x) + C_1 y_1(x) + \ldots + C_n y_n(x),$$

where y_p is any solution (called a **particular solution**) and $C_1 y_1(x) + \ldots + C_n y_n(x)$ is the general solution of the corresponding homogeneous DE.

Why? Suppose we have a single solution y_p (called a **particular solution**) of the inhomogeneous DE $Ly = Q(x)$.
Let y_h be the general solution of the homogeneous DE $Ly = 0$.
Then $L(y_p + y_h) = \underbrace{Ly_p}_{=Q(x)} + \underbrace{Ly_h}_{=0} = Q(x)$.
In other words, $y_p + y_h$ solves $Ly = Q(x)$ as well. Indeed it must be the general solution (note that it has the appropriate number of degrees of freedom).

Comment. If $y_p^{(1)}$ and $y_p^{(2)}$ are two solutions of $Ly = Q(x)$. What can you say about $y_p^{(1)} - y_p^{(2)}$?

[This difference should solve the homogeneous DE $Ly = 0$. Indeed, $L\left(y_p^{(1)} - y_p^{(2)}\right) = \underbrace{Ly_p^{(1)}}_{=Q(x)} - \underbrace{Ly_p^{(2)}}_{=Q(x)} = 0$.]

Example 76. (preview) Determine the general solution of $y'' + 4y = 12x$. *Hint*: $3x$ is a solution.

Solution. Here, $p(D) = D^2 + 4$. Because of the hint, we know that a particular solution is $y_p = 3x$.
The homogeneous DE $p(D)y = 0$ has solutions $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$. [Make sure this is clear!]
Therefore, the general solution to the original DE is $y_p + C_1 y_1 + C_2 y_2 = 3x + C_1\cos(2x) + C_2\sin(2x)$.

Next. How to find the particular solution $y_p = 3x$ ourselves.

The method of undetermined coefficients

The method of undetermined coefficients allows us to solve any inhomogeneous linear DE $Ly = Q(x)$ with constant coefficients if $Q(x)$ is a polynomial times an exponential (or a linear combination of such terms).

More precisely, $Q(x)$ needs to be a solution of a homogeneous linear DE with constant coefficients.

Example 77. Determine the general solution of $y'' + 4y = 12x$.

Solution. Here, $p(D) = D^2 + 4$, which has roots $\pm 2i$.
Hence, the general solution is $y(x) = y_p(x) + C_1\cos(2x) + C_2\sin(2x)$. It remains to find a particular solution y_p.
Noting that $D^2 \cdot (12x) = 0$, we apply D^2 to both sides of the DE.
We get $D^2(D^2 + 4) \cdot y = 0$, which is a homogeneous linear DE! Its general solution is $C_1 + C_2 x + C_3\cos(2x) + C_4\sin(2x)$. In particular, y_p is of this form for some choice of $C_1, \ldots, C_4$.
It simplifies our life to note that there has to be a particular solution of the simpler form $y_p = C_1 + C_2 x$.
[Why?! Because we know that $C_3\cos(2x) + C_4\sin(2x)$ can be added to any particular solution.]
It only remains to find appropriate values C_1, C_2 such that $y_p'' + 4y_p = 12x$. Since $y_p'' + 4y_p = 4C_1 + 4C_2 x$, comparing coefficients yields $4C_1 = 0$ and $4C_2 = 12$, so that $C_1 = 0$ and $C_2 = 3$. In other words, $y_p = 3x$.
Therefore, the general solution to the original DE is $y(x) = 3x + C_1\cos(2x) + C_2\sin(2x)$.

Example 78. (review) Determine the general solution of $y'' + 4y = 12x$.

Solution. The DE is $p(D)y = 12x$ with $p(D) = D^2 + 4$, which has roots $\pm 2i$. Thus, the general solution is $y(x) = y_p(x) + C_1\cos(2x) + C_2\sin(2x)$. It remains to find a particular solution y_p.

Since $D^2 \cdot (12x) = 0$, we apply D^2 to both sides of the DE to get the **homogeneous** DE $D^2(D^2+4) \cdot y = 0$.

Its general solution is $C_1 + C_2x + C_3\cos(2x) + C_4\sin(2x)$ and y_p must be of this form. Indeed, there must be a particular solution of the simpler form $y_p = C_1 + C_2x$ (because $C_3\cos(2x) + C_4\sin(2x)$ can be added to any y_p).

It remains to find appropriate values C_1, C_2 such that $y_p'' + 4y_p = 12x$. Since $y_p'' + 4y_p = 4C_1 + 4C_2x$, comparing coefficients yields $4C_1 = 0$ and $4C_2 = 12$, so that $C_1 = 0$ and $C_2 = 3$. In other words, $y_p = 3x$.

Therefore, the general solution to the original DE is $y(x) = 3x + C_1\cos(2x) + C_2\sin(2x)$.

Example 79. Determine the general solution of $y'' + 4y' + 4y = e^{3x}$.

Solution. The DE is $p(D)y = e^{3x}$ with $p(D) = D^2 + 4D + 4 = (D+2)^2$, which has roots $-2, -2$. Thus, the general solution is $y(x) = y_p(x) + (C_1 + C_2x)e^{-2x}$. It remains to find a particular solution y_p.

Since $(D-3)e^{3x} = 0$, we apply $(D-3)$ to the DE to get the **homogeneous** DE $(D-3)(D+2)^2y = 0$.

Its general solution is $(C_1 + C_2x)e^{-2x} + C_3e^{3x}$ and y_p must be of this form. Indeed, there must be a particular solution of the simpler form $y_p = Ce^{3x}$.

To determine the value of C, we plug into the original DE: $y_p'' + 4y_p' + 4y_p = (9 + 4 \cdot 3 + 4)Ce^{3x} \overset{!}{=} e^{3x}$. Hence, $C = 1/25$. Therefore, the general solution to the original DE is $y(x) = (C_1 + C_2x)e^{-2x} + \frac{1}{25}e^{3x}$.

We found a recipe for solving nonhomogeneous linear DEs with constant coefficients.

Our approach works for $p(D)y = f(x)$ whenever the right-hand side $f(x)$ is the solution of some homogeneous linear DE with constant coefficients: $q(D)f(x) = 0$

Theorem 80. (method of undetermined coefficients) To find a particular solution y_p to an inhomogeneous linear DE with constant coefficients $p(D)y = f(x)$:

- Find $q(D)$ so that $q(D)f(x) = 0$. [This does not work for all $f(x)$.]
- Let $r_1, \ldots, r_n$ be the ("old") roots of the polynomial $p(D)$.
  Let $s_1, \ldots, s_m$ be the ("new") roots of the polynomial $q(D)$.
- It follows that y_p solves the **homogeneous** DE $q(D)\,p(D)y = 0$.
  The characteristic polynomial of this DE has roots $r_1, \ldots, r_n, s_1, \ldots, s_m$.
  Let $v_1, \ldots, v_m$ be the "new" solutions (i.e. not solutions of the "old" $p(D)y = 0$).
  By plugging into $p(D)y_p = f(x)$, we find (unique) C_i so that $y_p = C_1v_1 + \ldots + C_mv_m$.

Because of the final step, this approach is often called **method of undetermined coefficients**.

For which $f(x)$ does this work? By Theorem 64, we know exactly which $f(x)$ are solutions to homogeneous linear DEs with constant coefficients: these are linear combinations of exponentials x^je^{rx} (which includes $x^j\,e^{ax}\cos(bx)$ and $x^j\,e^{ax}\sin(bx)$).

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Example 81. (again) Determine the general solution of $y'' + 4y' + 4y = e^{3x}$.

Solution. The "old" roots are $-2, -2$. The "new" roots are 3. Hence, there has to be a particular solution of the form $y_p = Ce^{3x}$. To find the value of C, we plug into the DE.

$y_p'' + 4y_p' + 4y_p = (9 + 4 \cdot 3 + 4)Ce^{3x} \overset{!}{=} e^{3x}$. Hence, $C = 1/25$.

Therefore, the general solution is $y(x) = (C_1 + C_2 x)e^{-2x} + \frac{1}{25}e^{3x}$.

Example 82. Determine the general solution of $y'' + 4y' + 4y = 7e^{-2x}$.

Solution. The "old" roots are $-2, -2$. The "new" roots are -2. Hence, there has to be a particular solution of the form $y_p = Cx^2e^{-2x}$. To find the value of C, we plug into the DE.

$y_p' = C(-2x^2 + 2x)e^{-2x}$

$y_p'' = C(4x^2 - 8x + 2)e^{-2x}$

$y_p'' + 4y_p' + 4y_p = 2Ce^{-2x} \overset{!}{=} 7e^{-2x}$

It follows that $C = 7/2$, so that $y_p = \frac{7}{2}x^2e^{-2x}$. The general solution is $y(x) = \left(C_1 + C_2 x + \frac{7}{2}x^2\right)e^{-2x}$.

Example 83. Determine a particular solution of $y'' + 4y' + 4y = 2e^{3x} - 5e^{-2x}$.

Solution. Write the DE as $Ly = 2e^{3x} - 5e^{-2x}$ where $L = D^2 + 4D + 4$. Instead of starting all over, recall that in Example 81 we found that $y_1 = \frac{1}{25}e^{3x}$ satisfies $Ly_1 = e^{3x}$. Also, in Example 82 we found that $y_2 = \frac{7}{2}x^2e^{-2x}$ satisfies $Ly_2 = 7e^{-2x}$.

By linearity, it follows that $L(Ay_1 + By_2) = ALy_1 + BLy_2 = Ae^{3x} + 7Be^{-2x}$.

To get a particular solution y_p of our DE, we need $A = 2$ and $7B = -5$.

Hence, $y_p = 2y_1 - \frac{5}{7}y_2 = \frac{2}{25}e^{3x} - \frac{5}{2}x^2e^{-2x}$.

Example 84. (homework) Determine the general solution of $y'' - 2y' + y = 5\sin(3x)$.

Solution. Since $D^2 - 2D + 1 = (D-1)^2$, the "old" roots are $1, 1$. The "new" roots are $\pm 3i$. Hence, there has to be a particular solution of the form $y_p = A\cos(3x) + B\sin(3x)$.

To find the values of A and B, we plug into the DE.

$y_p' = -3A\sin(3x) + 3B\cos(3x)$

$y_p'' = -9A\cos(3x) - 9B\sin(3x)$

$y_p'' - 2y_p' + y_p = (-8A - 6B)\cos(3x) + (6A - 8B)\sin(3x) \overset{!}{=} 5\sin(3x)$

Equating the coefficients of $\cos(x)$, $\sin(x)$, we obtain the two equations $-8A - 6B = 0$ and $6A - 8B = 5$.

Solving these, we find $A = \frac{3}{10}$, $B = -\frac{2}{5}$. Accordingly, a particular solution is $y_p = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x)$.

The general solution is $y(x) = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x) + (C_1 + C_2x)e^x$.

Example 85. (homework) What is the shape of a particular solution of $y'' + 4y' + 4y = x\cos(x)$?

Solution. The "old" roots are $-2, -2$. The "new" roots are $\pm i, \pm i$. Hence, there has to be a particular solution of the form $y_p = (C_1 + C_2x)\cos(x) + (C_3 + C_4x)\sin(x)$.

Continuing to find a particular solution. To find the value of the C_j's, we plug into the DE.

$y_p' = (C_2 + C_3 + C_4x)\cos(x) + (C_4 - C_1 - C_2x)\sin(x)$

$y_p'' = (2C_4 - C_1 - C_2x)\cos(x) + (-2C_2 - C_3 - C_4x)\sin(x)$

$y_p'' + 4y_p' + 4y_p = (3C_1 + 4C_2 + 4C_3 + 2C_4 + (3C_2 + 4C_4)x)\cos(x)$
$\quad + (-4C_1 - 2C_2 + 3C_3 + 4C_4 + (-4C_2 + 3C_4)x)\sin(x) \overset{!}{=} x\cos(x)$.

Equating the coefficients of $\cos(x)$, $x\cos(x)$, $\sin(x)$, $x\sin(x)$, we get the equations $3C_1 + 4C_2 + 4C_3 + 2C_4 = 0$, $3C_2 + 4C_4 = 1$, $-4C_1 - 2C_2 + 3C_3 + 4C_4 = 0$, $-4C_2 + 3C_4 = 0$.

Solving (this is tedious!), we find $C_1 = -\frac{4}{125}$, $C_2 = \frac{3}{25}$, $C_3 = -\frac{22}{125}$, $C_4 = \frac{4}{25}$.

Hence, $y_p = \left(-\frac{4}{125} + \frac{3}{25}x\right)\cos(x) + \left(-\frac{22}{125} + \frac{4}{25}x\right)\sin(x)$.

Example 86. (homework) What is the shape of a particular solution of $y'' + 4y' + 4y = 4e^{3x}\sin(2x) - x\sin(x)$.

Solution. The "old" roots are $-2, -2$. The "new" roots are $3 \pm 2i, \pm i, \pm i$.

Hence, there has to be a particular solution of the form

$y_p = C_1e^{3x}\cos(2x) + C_2e^{3x}\sin(2x) + (C_3 + C_4x)\cos(x) + (C_5 + C_6x)\sin(x)$.

Continuing to find a particular solution. To find the values of $C_1, \ldots, C_6$, we plug into the DE. But this final step is so boring that we don't go through it here. Computers (currently?) cannot afford to be as selective; mine obediently calculated: $y_p = -\frac{4}{841}e^{3x}(20\cos(2x) - 21\sin(2x)) + \frac{1}{125}((-22 + 20x)\cos(x) + (4 - 15x)\sin(x))$

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A more general method for finding particular solutions: variation of parameters

Review. To find the general solution of an inhomogeneous linear DE $Ly = Q(x)$, we only need to find a single **particular solution** y_p. Then the general solution is $y_p + y_h$, where y_h is the general solution of $Ly = 0$.

Theorem 87. (variation of parameters) A particular solution to the inhomogeneous second-order linear DE $Ly = y'' + P_1(x)y' + P_0(x)y = Q(x)$ is given by:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x), \quad u_1(x) = -\int \frac{y_2(x)Q(x)}{W(x)}\mathrm{d}x, \quad u_2(x) = \int \frac{y_1(x)Q(x)}{W(x)}\mathrm{d}x,$$

where y_1, y_2 are independent solutions of $Ly = 0$ and $W = y_1y_2' - y_1' y_2$ is their Wronskian.

Comment. We obtain the general solution if we consider all possible constants of integration in the formula for y_p.

Proof. Let us look for a particular solution of the form $y_p = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$.
This "ansatz" is called **variation of constants/parameters**. We plug into the DE to determine conditions on u_1, u_2 so that y_p is a solution. The DE will give us one condition and (since there are two unknowns), it is reasonable to expect that we can impose a second condition (labelled below as "our wish") to make our life simpler.

We compute $y_p' = \underbrace{u_1'y_1 + u_2'y_2}_{=0 \text{ (our wish)}} + u_1y_1' + u_2y_2'$ and, thus, $y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''$.

["Our wish" was chosen so that y_p'' would only involve first derivatives of u_1 and u_2.]

Therefore, plugging into the DE results in

$$Ly_p = u_1'y_1' + u_2'y_2' + \underbrace{u_1y_1'' + u_2y_2'' + P_1(x)(u_1y_1' + u_2y_2') + P_0(x)(u_1y_1 + u_2y_2)}_{=u_1Ly_1 + u_2Ly_2 = 0} \overset{!}{=} Q(x).$$

We conclude that y_p solves the DE if the following two conditions (the first is "our wish") are satisfied:

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0, \\ u_1'y_1' + u_2'y_2' &= Q(x). \end{aligned}$$

These are linear equations in u_1' and u_2'. Solving gives $u_1' = \frac{-y_2\,Q(x)}{y_1y_2' - y_1'\,y_2}$ and $u_2' = \frac{y_1\,Q(x)}{y_1y_2' - y_1'\,y_2}$, and it only remains to integrate. □

Comment. In matrix-vector form, the equations are $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ Q(x) \end{bmatrix}$.

Our solution then follows from multiplying $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} = \frac{1}{y_1y_2' - y_1'\,y_2}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}$ with $\begin{bmatrix} 0 \\ Q(x) \end{bmatrix}$.

Advanced comment. $W = y_1y_2' - y_1'\,y_2$ is called the **Wronskian** of y_1 and y_2. In general, given a linear homogeneous DE of order n with solutions $y_1, \ldots, y_n$, the Wronskian of $y_1, \ldots, y_n$ is the determinant of the matrix where each column consists of the derivatives of one of the y_i. One useful property of the Wronskian is that it is nonzero if and only if the $y_1, \ldots, y_n$ are linearly independent and therefore generate the general solution.

Example 88. Determine the general solution of $y'' - 2y' + y = \frac{e^x}{x}$.

Solution. This DE is of the form $Ly = Q(x)$ with $L = D^2 - 2D + 1$ and $Q(x) = \frac{e^x}{x}$.
Since $L = (D-1)^2$, the homogeneous DE has the two solutions $y_1 = e^x$, $y_2 = xe^x$.
The corresponding Wronskian is $W = y_1y_2' - y_1'\,y_2 = e^{2x}$.
By variation of parameters (Theorem 87), we find that a particular solution is

$$y_p = -y_1\int \frac{y_2Q}{W}\mathrm{d}x + y_2\int \frac{y_1Q}{W}\mathrm{d}x = -e^x\int 1\mathrm{d}x + xe^x\int \frac{1}{x}\mathrm{d}x = xe^x(\ln|x| - 1).$$

The general solution therefore is $xe^x(\ln|x| - 1) + (C_1 + C_2x)e^x$.
Comment. Adding constants of integration in the formula for y_p, we get $-e^x(x + D_1) + xe^x(\ln|x| + D_2)$, which is the general solution. Any choice of constants suffices to give us a particular solution.

Systems of differential equations

Modeling two connected fluid tanks

Example 89. Consider two brine tanks. Tank T_1 contains 24gal water containing 3lb salt, and tank T_2 contains 9gal pure water.

- T_1 is being filled with 54gal/min water containing 0.5lb/gal salt.
- 72gal/min well-mixed solution flows out of T_1 into T_2.
- 18gal/min well-mixed solution flows out of T_2 into T_1.
- Finally, 54gal/min well-mixed solution is leaving T_2.

How much salt is in the tanks after t minutes?

Solution. Note that the amount of water in each tank is constant because the flows balance each other.
Let $y_i(t)$ denote the amount of salt (in lb) in tank T_i after time t (in min). In the time interval $[t, t+\Delta t]$:
$\Delta y_1 \approx 54 \cdot \frac{1}{2} \cdot \Delta t - 72 \cdot \frac{y_1}{24} \cdot \Delta t + 18 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_1' = 27 - 3y_1 + 2y_2$. Also, $y_1(0) = 3$.
$\Delta y_2 \approx 72 \cdot \frac{y_1}{24} \cdot \Delta t - 72 \cdot \frac{y_2}{9} \cdot \Delta t$, so $y_2' = 3y_1 - 8y_2$. Also, $y_2(0) = 0$.

One strategy to solve this system is to first combine the two DEs to get a single equation for y_1.

- From the first DE, we get $y_2 = \frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2}$.
- Using this in the second DE, we obtain $\left(\frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2}\right)' = 3y_1 - 8\left(\frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2}\right)$.
  Simplified, this is $y_1'' + 11y_1' + 18y_1 = 216$.
- We already have the initial condition $y_1(0) = 3$. We get a second one by combining $y_2 = \frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2}$ with $y_2(0) = 0$ to get $0 = y_2(0) = \frac{1}{2}y_1'(0) + \frac{3}{2}y_1(0) - \frac{27}{2} = \frac{1}{2}y_1'(0) - 9$, which simplifies to $y_1'(0) = 18$.
- The IVP $y_1'' + 11y_1' + 18y_1 = 216$ with initial conditions $y_1(0) = 3$ and $y_1'(0) = 18$ is one that we can solve!
  - general solution of the corresponding homogeneous equation: $y_h = C_1 e^{-2t} + C_2 e^{-9t}$
  - particular solution: $y_p = C$, $y_p = \frac{216}{18} = 12$
  - Hence, the general solution to the DE is $y(x) = 12 + C_1 e^{-2t} + C_2 e^{-9t}$
    $y(0) = 12 + C_1 + C_2 \overset{!}{=} 3$, $y'(0) = -2C_1 - 9C_2 \overset{!}{=} 18$
    Solution: $C_1 = -9$, $C_2 = 0$

  It has the unique solution $y_1(t) = 12 - 9e^{-2t}$.
- It follows that $y_2 = \frac{1}{2}y_1' + \frac{3}{2}y_1 - \frac{27}{2} = \frac{9}{2} - \frac{9}{2}e^{-2t}$.

Note. We could have found a particular solution with less calculations by observing (looking at "old" and "new" roots) that there must be a solution of the form $\boldsymbol{y}_p(t) = \boldsymbol{a}$. We can then find $\boldsymbol{a}$ by plugging into the differential equation. However, noticing that, for a constant solution, each tank has to have a constant concentration of 0.5lb/gal of salt, we find $\boldsymbol{y}_p(t) = \begin{bmatrix} 12 \\ 4.5 \end{bmatrix}$.

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Example 90. Write the system of equations

$$\begin{array}{rclr} y_1' & = & -3y_1 + 2y_2 + 27 & y_1(0) = 3 \\ y_2' & = & 3y_1 - 8y_2 & y_2(0) = 0 \end{array}$$

from the previous problem in matrix-vector notation.

Solution. If we write $\boldsymbol{y} = \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]$, then the system becomes

$$\boldsymbol{y}' = \begin{bmatrix} -3 & 2 \\ 3 & -8 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 27 \\ 0 \end{bmatrix}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}.$$

Advanced comment. We only use the matrix-vector notation as a device for writing the system of equations in a more compact form. However, it turns out that the matrix-vector notation makes certain techniques more transparent (just like writing a system of equations in the form $A\boldsymbol{x} = \boldsymbol{b}$ suggests introducing the matrix inverse to simply write $\boldsymbol{x} = A^{-1}\boldsymbol{b}$). For instance, the unique solution to a homogeneous linear system $\boldsymbol{y}' = M\boldsymbol{y}$ (where M is a matrix with constant entries) with initial condition $\boldsymbol{y}(0) = \boldsymbol{c}$ can be expressed as $\boldsymbol{y}(x) = e^{Mx}\boldsymbol{c}$, just as in the case of a single linear DE. Here, e^{Mx} is the **matrix exponential**. This will be one of the topics discussed in both Differential Equations II and Linear Algebra II.

Example 91. (extra) Three brine tanks T_1, T_2, T_3.

T_1 contains 20gal water with 10lb salt, T_2 40gal pure water, T_3 50gal water with 30lb salt.

T_1 is filled with 10gal/min water with 2lb/gal salt. 10gal/min well-mixed solution flows out of T_1 into T_2. Also, 10gal/min well-mixed solution flows out of T_2 into T_3. Finally, 10gal/min well-mixed solution is leaving T_3. How much salt is in the tanks after t minutes?

Solution. Let $y_i(t)$ denote the amount of salt (in lb) in tank T_i after time t (in min).

In the time interval $[t, t + \Delta t]$:

$\Delta y_1 \approx 10 \cdot 2 \cdot \Delta t - 10 \frac{y_1}{20} \cdot \Delta t$, so $y_1' = 20 - \frac{1}{2} y_1$. Also, $y_1(0) = 10$.

$\Delta y_2 \approx 10 \cdot \frac{y_1}{20} \cdot \Delta t - 10 \frac{y_2}{40} \cdot \Delta t$, so $y_2' = \frac{1}{2} y_1 - \frac{1}{4} y_2$. Also, $y_2(0) = 0$.

$\Delta y_3 \approx 10 \cdot \frac{y_2}{40} \cdot \Delta t - 10 \frac{y_3}{50} \cdot \Delta t$, so $y_3' = \frac{1}{4} y_2 - \frac{1}{5} y_3$. Also, $y_3(0) = 30$.

Using matrix notation and writing $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, this is $\boldsymbol{y}' = \begin{bmatrix} -1/2 & 0 & 0 \\ 1/2 & -1/4 & 0 \\ 0 & 1/4 & -1/5 \end{bmatrix} \boldsymbol{y} + \begin{bmatrix} 20 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{y}(0) = \begin{bmatrix} 10 \\ 0 \\ 30 \end{bmatrix}$.

We can solve this IVP!

[Do it! Start by finding y_1 from the first DE, then move on to $y_2 \ldots$]

Here, we content ourselves with finding a particular solution (and ignoring the initial conditions). The method of undetermined coefficients tells us that there is a solution of the form $\boldsymbol{y}_p(t) = \boldsymbol{a}$. Of course, we can find $\boldsymbol{a}$ by plugging into the differential equation. However, noticing that, for a constant solution, each tank has to have a concentration of 2lb/gal of salt, we find $\boldsymbol{y}_p = (40, 80, 100)$ without calculation.

Example 92.

(a) Determine the general solution to $y_1' = 5y_1 + 4y_2$, $y_2' = 8y_1 + y_2$.

Comment. In matrix form, with $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, this is $\boldsymbol{y}' = \begin{bmatrix} 5 & 4 \\ 8 & 1 \end{bmatrix} \boldsymbol{y}$.

(b) Solve the IVP $y_1' = 5y_1 + 4y_2$, $y_2' = 8y_1 + y_2$, $y_1(0) = 0$, $y_2(0) = 1$.

(c) Determine a particular solution to $y_1' = 5y_1 + 4y_2 + e^{2x}$, $y_2' = 8y_1 + y_2$.

(d) Determine the general solution to $y_1' = 5y_1 + 4y_2 + e^{2x}$, $y_2' = 8y_1 + y_2$.

Solution.

(a) Since $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1$ (from the first equation), we have $y_2' = \frac{1}{4}y_1'' - \frac{5}{4}y_1'$.
Using these in the second equation, we get $\frac{1}{4}y_1'' - \frac{5}{4}y_1' = 8y_1 + \frac{1}{4}y_1' - \frac{5}{4}y_1$.
Simplified, this is $y_1'' - 6y_1' - 27y_1 = 0$.
This is a homogeneous linear DE with constant coefficients. The characteristic roots are $-3, 9$.
We therefore obtain $y_1 = C_1 e^{-3x} + C_2 e^{9x}$ as the general solution.
Thus, $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1 = \frac{1}{4}(-3C_1 e^{-3x} + 9C_2 e^{9x}) - \frac{5}{4}(C_1 e^{-3x} + C_2 e^{9x}) = -2C_1 e^{-3x} + C_2 e^{9x}$.

(b) We already have the general solutions y_1, y_2 to the two DEs. We need to determine the (unique) values of C_1 and C_2 to match the initial conditions:
$y_1(0) = C_1 + C_2 \overset{!}{=} 0$
$y_2(0) = -2C_1 + C_2 \overset{!}{=} 1$
We solve these two equations and find $C_1 = -\frac{1}{3}$ and $C_2 = \frac{1}{3}$.
The unique solution to the IVP therefore is $y_1 = -\frac{1}{3}e^{-3x} + \frac{1}{3}e^{9x}$ and $y_2 = \frac{2}{3}e^{-3x} + \frac{1}{3}e^{9x}$.

(c) We proceed as in the first part to write $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1 - \frac{1}{4}e^{2x}$.
Using this in the second equation, we get $\frac{1}{4}y_1'' - \frac{5}{4}y_1' - \frac{1}{2}e^{2x} = 8y_1 + \frac{1}{4}y_1' - \frac{5}{4}y_1 - \frac{1}{4}e^{2x}$.
Simplified, this is $y_1'' - 6y_1' - 27y_1 = e^{2x}$.
This is an inhomogeneous linear DE with constant coefficients. Since the "old" roots are $-3, 9$, while the "new" root is 2, there must a particular solution of the form $y_1 = Ce^{2x}$. Plugging this y_1 into the DE, we get $y_1'' - 6y_1' - 27y_1 = (4 - 6 \cdot 2 - 27)Ce^{2x} = -35Ce^{2x} \overset{!}{=} e^{2x}$. Hence, $C = -\frac{1}{35}$.
Corresponding to $y_1 = -\frac{1}{35}e^{2x}$ we get $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1 = -\frac{1}{35}\left(\frac{1}{4} \cdot 2 - \frac{5}{4}\right)e^{2x} = \frac{3}{35}e^{2x}$.

(d) The general solution is the particular solution (previous part) plus the general solution to the homogeneous system (first part):
$y_1 = -\frac{1}{35}e^{2x} + C_1 e^{-3x} + C_2 e^{9x}$
$y_2 = \frac{3}{35}e^{2x} - 2C_1 e^{-3x} + C_2 e^{9x}$

Higher-order linear DEs as first-order systems

Example 93. Write the (second-order) differential equation $y'' = 2y' + 5y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$ and $y_2 = y'$. Then $y'' = 2y' + 5y$ becomes $y_2' = 2y_2 + 5y_1$.

Therefore, $y'' = 2y' + 5y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = 5y_1 + 2y_2 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 5 & 2 \end{bmatrix} \boldsymbol{y}$.

Comment. This illustrates why we might care about systems of DEs, even if we work with only one function.

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Example 94. Write the (third-order) differential equation $y''' = 3y'' - 2y' + 4y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' = 3y'' - 2y' + 4y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = 4y_1 - 2y_2 + 3y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -2 & 3 \end{bmatrix} \boldsymbol{y}$.

Example 95. Consider the following system of (second-order) initial value problems:

$$\begin{array}{l} y_1'' = 2y_1' - 3y_2' + 7y_2 \\ y_2'' = 4y_1' + y_2' - 5y_1 \end{array} \qquad y_1(0) = 2, \; y_1'(0) = 3, \; y_2(0) = -1, \; y_2'(0) = 1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

Extra: Two more applications of systems of DEs

Example 96. (military strategy) Lanchester's equations model two opposing forces during "aimed fire" battle.

Let $x(t)$ and $y(t)$ describe the number of troops on each side. Then Lanchester (during World War I) assumed that the rates $-x'(t)$ and $-y'(t)$, at which soldiers are put out of action, are proportional to the number of opposing forces. That is:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -\alpha y(t) \\ -\beta x(t) \end{bmatrix}, \quad \text{or, in matrix form: } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -\alpha \\ -\beta & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The proportionality constants $\alpha, \beta > 0$ indicate the strength of the forces ("fighting effectiveness coefficients").

These are simple linear DEs with constant coefficients, which we have learned how to solve.

Comment. The "aimed fire" means that all combatants are engaged, as is common in modern combat with long-range weapons. This is rather different than ancient combat where soldiers were engaging one opponent at a time.

For more details, see: `https://en.wikipedia.org/wiki/Lanchester%27s_laws`

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Example 97. (epidemiology) Let us indicate the popular SIR model for short outbreaks of diseases among a population of constant size N.

In a SIR model, the population is compartmentalized into $S(t)$ susceptible, $I(t)$ infected and $R(t)$ recovered (or resistant) individuals ($N = S(t) + I(t) + R(t)$). In the Kermack-McKendrick model, the outbreak of a disease is modeled by

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I,$$

with γ modeling the recovery rate and β the infection rate. Note that this is a non-linear system of differential equations. For more details and many variations used in epidemiology, see:

https://en.wikipedia.org/wiki/Compartmental_models_in_epidemiology

Comment. The following variation

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I R, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I R,$$

which assumes "infectious recovery", was used in 2014 to predict that facebook might lose 80% of its users by 2017. It is that claim, not mathematics (or even the modeling), which attracted a lot of media attention.

http://blogs.wsj.com/digits/2014/01/22/controversial-paper-predicts-facebook-decline/

A closer look at second-order linear DEs

Application: motion of a mass on a spring

Example 98. The motion of a mass m attached to a spring is described by

$$m y'' + k y = 0$$

where y is the displacement from the equilibrium position and $k > 0$ is the spring constant.

Why? This follows from Hooke's law $F = -k y$ combined with Newton's second law $F = m a = m y''$. (Note that the minus sign is needed because the force on the mass is in direction opposite to the displacement.)

Comment. By measuring y as the displacement from equilibrium, it doesn't matter whether the mass is attached horizontally or vertically (gravity is taken into account by the extra stretch in the spring due to the mass).

Solving this DE, we find that the general solution is

$$y(t) = A\cos(\omega t) + B\sin(\omega t)$$

where $\omega = \sqrt{k/m}$ (note that the characteristic roots are $\pm i\sqrt{\frac{k}{m}}$). We observe that:

- The motion $y(t)$ is periodic with **period** $2\pi/\omega$.

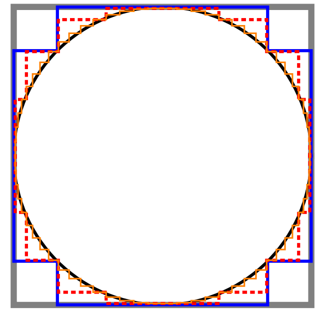
  This follows from the fact that both $\cos(t)$ and $\sin(t)$ have period 2π.

- The **amplitude** of the motion $y(t)$ is $\sqrt{A^2 + B^2}$.

  This follows from the fact that $y(t) = A\cos(\omega t) + B\sin(\omega t) = r\cos(\omega t - \alpha)$ where (r, α) are the **polar coordinates** for (A, B). In particular, the amplitude is $r = \sqrt{A^2 + B^2}$.

  Can you explain the reason for being able to write $y(t)$ as $r\cos(\omega t - \alpha)$ using DEs? We will do so next time.

(A Halloween scare!) π is the perimeter of a circle enclosed in a square with edge length 1. The perimeter of the square is 4, which approximates π. To get a better approximation, we "fold" the vertices of the square towards the circle (and get the blue polygon). This construction can be repeated for even better approximations and, in the limit, our shape will converge to the true circle. At each step, the perimeter is 4, so we conclude that $\pi = 4$, contrary to popular belief.



Can you pin-point the fallacy in this argument?

(We are not doing something completely silly! For instance, the areas of our approximations do converge to $\pi / 4$, the area of the circle.)

The "solution" is below...

($\pi = 4$, "solution")

We are constructing curves c_n with the property that $c_n \to c$ where c is the circle. This convergence can be understood, for instance, in the same sense $\|c_n - c\| \to 0$ with the norm measuring the maximum distance between the two curves.

Since $c_n \to c$ we then wanted to conclude that $\text{perimeter}(c_n) \to \text{perimeter}(c)$, leading to $4 \to \pi$.

However, in order to conclude from $x_n \to x$ that $f(x_n) \to f(x)$ we need that f is continuous (at x)!!

The "function" perimeter, however, is not continuous. In words, this means that (as we see in this example) curves can be arbitrarily close, yet have very different arc length.

We can dig a little deeper: as we learned in Calculus II, the arc length of a function $y = f_n(x)$ for $x \in [a, b]$ is

$$\int_a^b \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \int_a^b \sqrt{1 + f_n'(x)^2}\mathrm{d}x.$$

Observe that this involves $f_n'(x)$. Try to see why the operator D that sends f to f' is not continuous with respect to the distance induced by the norm

$$\|f\| = \left(\int_a^b f(x)^2 \mathrm{d}x\right)^{1/2}.$$

In words, two functions f and g can be arbitrarily close, yet have very different derivatives f' and g'.

That's a huge issue in **functional analysis**, which is the generalization of linear algebra to infinite dimensional spaces (like the space of all differentiable functions). The linear operators ("matrices") on these spaces frequently fail to be continuous.

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The amplitude of oscillations

Example 99. If (r,α) are the **polar coordinates** for (A,B), then

$$A\cos(\omega t)+B\sin(\omega t)=r\cos(\omega t-\alpha).$$

In particular, $A\cos(\omega t)+B\sin(\omega t)$ is periodic with **amplitude** $r=\sqrt{A^2+B^2}$.

ω is the (circular) **frequency** and α is called the **phase angle**.

Why? Both sides solve

$$y''+\omega y=0.$$

The LHS has initial values $y(0)=A$ and $y'(0)=\omega B$, the RHS has $y(0)=r\cos(\alpha)$ and $y'(0)=r\omega\sin(\alpha)$. Hence, the two are equal if $A=r\cos(\alpha)$ and $B=r\sin(\alpha)$.

Alternatively. If you like trig identities, this follows from:
$A\cos(\omega t)+B\sin(\omega t)=r(\cos(\alpha)\cos(\omega t)+\sin(\alpha)\sin(\omega t))=r\cos(\omega t-\alpha)$.

Review. How to calculate the polar coordinates (r,α) for (A,B)?

We need to find $r\geqslant 0$ and $\alpha\in[0,2\pi)$ such that $(A,B)=r(\cos\alpha,\sin\alpha)$.

Hence, $r=\sqrt{A^2+B^2}$ and α is determined by $\cos(\alpha)=\frac{A}{r}$ and $\sin(\alpha)=\frac{B}{r}$.

In particular, $\tan(\alpha)=\frac{B}{A}$ and, if careful, we can compute α using $\tan^{-1}$ as

$$\alpha=\tan^{-1}\left(\frac{B}{A}\right)+\begin{cases}0, & \text{if } (A,B) \text{ in first quadrant,}\\ 2\pi, & \text{if } (A,B) \text{ in fourth quadrant,}\\ \pi, & \text{otherwise.}\end{cases}$$

Example 100. What is the period and the amplitude of the oscillations $\cos(4t)-3\sin(4t)$?

Solution. The period is $\frac{2\pi}{4}=\frac{\pi}{2}$.

The amplitude is $\sqrt{1^2+(-3)^2}=\sqrt{10}$.

Example 101. The motion of a mass on a spring is described by $5y''+2y=0$, $y(0)=3$, $y'(0)=-1$. What is the period and the amplitude of the resulting oscillations?

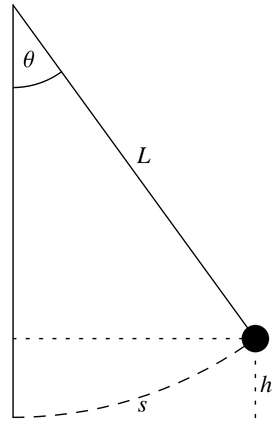
Solution. The characteristic roots are $\pm i\omega$ with $\omega=\sqrt{\frac{2}{5}}$. The general solution is $y(t)=A\cos(\omega t)+B\sin(\omega t)$.

The period of the oscillations therefore is $\frac{2\pi}{\omega}=2\pi\sqrt{\frac{5}{2}}=\pi\sqrt{10}$.

To meet the initial conditions, we need $y(0)=A\overset{!}{=}3$ and $y'(0)=\omega B\overset{!}{=}-1$. The latter implies $B=-\frac{1}{\omega}=-\sqrt{\frac{5}{2}}$.

Hence, the amplitude of the oscillations is $\sqrt{A^2+B^2}=\sqrt{3^2+\frac{5}{2}}=\sqrt{\frac{23}{2}}$.

Application: motion of a pendulum



Example 102. Show that the motion of an ideal pendulum is described by

$$L\theta'' + g\sin(\theta) = 0,$$

where θ is the angular displacement and L is the length of the pendulum.

And, as usual, g is acceleration due to gravity.

For short times and small angles, this motion is approximately described by

$$L\theta'' + g\theta = 0.$$

This is because, if θ is small, then $\sin(\theta) \approx \theta$. For instance, for $\theta = 15°$ the error $\theta - \sin\theta$ is about 1%.

Solution. (Newton's second law) The tangential component of the gravitational force is $F = -\sin\theta \cdot mg$. Combining this with Newton's second law, according to which $F = ma = mL\theta''$ (note that $a = s''$ where $s = L\theta$), we obtain the claimed DE.

Solution. (conservation of energy) Alternatively, we can use conservation of energy to derive the DE. Again, we assume the string to be massless, and let m be the swinging mass. Let s and h be as in the sketch above.

The velocity (more accurately, the speed) of the mass is $v = \frac{\mathrm{d}s}{\mathrm{d}t} = L\frac{\mathrm{d}\theta}{\mathrm{d}t}$.

Its kinetic energy therefore is $T = \frac{1}{2}mv^2 = \frac{1}{2}mL^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2$.

On the other hand, the potential energy is $V = mgh = mgL(1 - \cos\theta)$ (weight mg times height h).

By the principle of conservation of energy, the sum of these is constant: $T + V = \text{const}$

Taking the time derivative, this becomes $\frac{1}{2}mL^2 2\frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + mgL\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$. Cancelling terms, we obtain the DE.

Example 103. The motion of a pendulum is described by $\theta'' + 9\theta = 0$, $\theta(0) = 1/4$, $\theta'(0) = 0$. What is the period and the amplitude of the resulting oscillations?

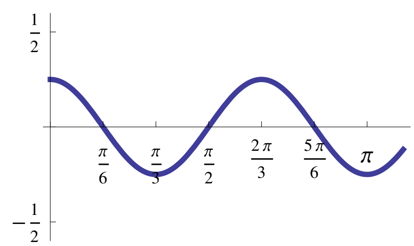
Solution. The roots of the characteristic polynomial are $\pm 3i$.

Hence, $\theta(t) = A\cos(3t) + B\sin(3t)$. $\theta(0) = A = 1/4$. $\theta'(0) = 3B = 0$.

Therefore, the solution is $\theta(t) = 1/4\cos(3t)$.

Hence, the period is $2\pi/3$ and the amplitude is $1/4$.

Comment. $1/4$ is about $14.3°$.



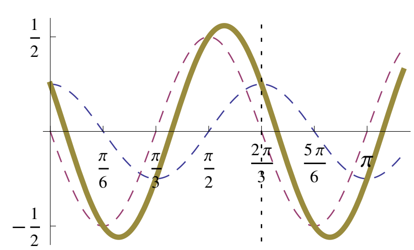
Example 104. The motion of a pendulum is described by $\theta'' + 9\theta = 0$, $\theta(0) = 1/4$, $\theta'(0) = -\frac{3}{2}$ ("initial kick"). What is the period and the amplitude of the resulting oscillations?

Solution. This time, $\theta(0) = A = 1/4$. $\theta'(0) = 3B = -3/2$.

Therefore, the solution is $\theta(t) = \frac{1}{4}\cos(3t) - \frac{1}{2}\sin(3t)$.

Hence, the period is $2\pi/3$ and the amplitude is $\sqrt{\frac{1}{4^2} + \frac{1}{2^2}} = \frac{\sqrt{5}}{4} \approx 0.559$.

Comment. Using polar coordinates, we get $\theta(t) = \frac{\sqrt{5}}{4}\cos(3t - \alpha)$ with phase angle $\alpha = \tan^{-1}(-2) + 2\pi \approx 5.176$.



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The qualitative effects of damping

The motion of a mass on a spring (or the approximate motion of a pendulum), with damping taken into account, can be modeled by the DE

$$y'' + dy' + cy = 0$$

with $c > 0$ and $d \geqslant 0$. The term dy' models damping (e.g. friction, air resistance) proportional to the velocity y'.

The characteristic equation $r^2 + dr + c = 0$ has roots $\frac{1}{2}\left(-d \pm \sqrt{d^2 - 4c}\right)$.

The nature of the solutions depends on whether the **discriminant** $\Delta = d^2 - 4c$ is positive, negative, or zero.

Undamped. $d = 0$. In that case, $\Delta < 0$. We get two complex roots $\pm i\omega$ with $\omega = \sqrt{c}$.

Solutions: $A\cos(\omega t) + B\sin(\omega t) = r\cos(\omega t - \alpha)$ where $(A, B) = r(\cos\alpha, \sin\alpha)$

These are oscillations with frequency ω and amplitude r.

Underdamped. $d > 0$, $\Delta < 0$. We get two complex roots $-\rho \pm i\omega$ with $-\rho = -d/2 < 0$.

Solutions: $e^{-\rho t}[A\cos(\omega t) + B\sin(\omega t)] = e^{-\rho t}[r\cos(\omega t - \alpha)]$ ($\to 0$ as $t \to \infty$)

These are oscillations with amplitude going to zero.

Critically damped. $d > 0$, $\Delta = 0$. We get one (double) real root $-\rho < 0$.

Solutions: $(A + Bt)\,e^{-\rho t}$ ($\to 0$ as $t \to \infty$)

There are no oscillations. (Can you see why we cross the t-axis at most once?)

Overdamped. $d > 0$, $\Delta > 0$. We get two real roots $-\rho_1, -\rho_2 < 0$. [negative because $c, d > 0$]

Solutions: $A e^{-\rho_1 t} + B e^{-\rho_2 t}$ ($\to 0$ as $t \to \infty$)

There are no oscillations. (Again, there is at most one crossing of the t-axis.)

Example 105. The motion of a mass on a spring is described by $5y'' + dy' + 2y = 0$. For which value of d is the motion critically damped?

Solution. The characteristic roots are $\frac{1}{2}\left(-d \pm \sqrt{d^2 - 40}\right)$. The motion is critically damped if $d^2 - 40 = 0$. Equivalently, if $d = \sqrt{40}$.

Adding external forces and the phenomenon of resonance

The motion of a mass on a spring, with an external force $f(t)$ taken into account, can be modeled by the DE

$$y'' + cy = f(t).$$

Example 106. Describe the solutions of $y'' + 4y = \cos(\lambda t)$.

Solution. The "old" roots are $\pm 2i$ so that 2 is the **natural frequency** (the frequency at which the system would oscillate in the absence of external forces; mathematically, this reflects the fact that the general solution to the corresponding homogeneous DE is $A\cos(2t) + B\sin(2t)$, which has frequency $\omega = 2$).

The "new" roots are $\pm\lambda i$ where λ is the **external frequency**.

Case 1: $\boldsymbol{\lambda \neq 2}$. Then there is a particular solution of the form $y_p = A\cos(\lambda t) + B\sin(\lambda t)$. To determine the unique values of A, B, we plug into the DE:

$$y_p'' + 4y_p = (4 - \lambda^2)A\cos(\lambda t) + (4 - \lambda^2)B\sin(\lambda t) \overset{!}{=} \cos(\lambda t)$$

We conclude that $(4 - \lambda^2)A = 1$ and $(4 - \lambda^2)B = 0$. Solving these, we find $A = 1/(4 - \lambda^2)$ and $B = 0$.

Thus, the general solution is of the form $y = \frac{1}{4 - \lambda^2}\cos(\lambda t) + C_1\cos(2t) + C_2\sin(2t)$.

Case 2: $\boldsymbol{\lambda = 2}$. Now, there is a particular solution of the form $y_p = At\cos(2t) + Bt\sin(2t)$. To determine the unique values of A, B, we again plug into the DE (which is more work this time):

$$y_p'' + 4y_p \overset{\text{work}}{=} 4B\cos(2t) - 4A\sin(2t) \overset{!}{=} \cos(2t)$$

We conclude that $4B = 1$ and $-4A = 0$. Solving these, we find $A = 0$ and $B = 1/4$.

Thus, the general solution is of the form $y = \frac{1}{4}t\sin(2t) + C_1\cos(2t) + C_2\sin(2t)$.

Note that the amplitude in y_p increases without bound (so that the same is true for the general solution). This phenomenon is called **resonance**; it occurs if an external frequency matches a natural frequency.

> If an external frequency matches a natural frequency, then **resonance** occurs.
>
> In that case, we obtain amplitudes that grow without bound.

Resonance (or anything close to it) is very important for practical purposes because large amplitudes can be very destructive: singing to shatter glass, earth quake waves and buildings, marching soldiers on bridges, ...

Comment. Mathematically speaking, the "old" and "new" roots overlap in an inhomogeneous linear DE. In that case, the solutions acquire a factor of the variable t (or x) which changes the nature of the solutions.

Example 107. Consider $y'' + 9y = 10\cos(2\lambda t)$. For what value of λ does resonance occur?

Solution. The natural frequency is 3. The external frequency is 2λ. Hence, resonance occurs when $\lambda = \frac{3}{2}$.

Example 108. The motion of a mass on a spring under an external force is described by $5y'' + 2y = -2\sin(3\lambda t)$. For which value of λ does resonance occur?

Solution. The natural frequency is $\sqrt{\frac{2}{5}}$. The external frequency is 3λ. Hence, resonance occurs when $\lambda = \frac{1}{3}\sqrt{\frac{2}{5}}$.

Example 109. The motion of a mass on a spring under an external force is described by $3y'' + ry = \cos(t/2)$. For which value of $r > 0$ does resonance occur?

Solution. The natural frequency is $\sqrt{\frac{r}{3}}$. The external frequency is $\frac{1}{2}$. Hence, resonance occurs when $\sqrt{\frac{r}{3}} = \frac{1}{2}$. This happens if $r = 3 \cdot \left(\frac{1}{2}\right)^2 = \frac{3}{4}$.

External forces plus damping (extra material)

Example 110. Find the general solution of $2y'' + 2y' + y = 10\sin(t)$.

Solution. The "old" roots are $\frac{1}{4}(-2 \pm \sqrt{4-8}) = -\frac{1}{2} \pm \frac{1}{2}i$.

Accordingly, the system without external force is underdamped. (Make sure that this is clear to you!)

Proceeding as usual, we find $y_p = -4\cos(t) - 2\sin(t) = \sqrt{20}(\cos(t-\alpha))$ with $\alpha = \tan^{-1}(1/2) + \pi \approx 3.605$.

Here, we used that $(-4, -2) = \sqrt{20}(\cos\alpha, \sin\alpha)$.

Hence, the general solution is $y(t) = \underbrace{\sqrt{20}\cos(t-\alpha)}_{y_{\text{sp}}} + \underbrace{e^{-t/2}\left(C_1\cos\left(\frac{t}{2}\right) + C_2\sin\left(\frac{t}{2}\right)\right)}_{y_{\text{tr}} \to 0 \text{ as } t\to\infty}$.

Observe how $y = y_{\text{tr}} + y_{\text{sp}}$ splits into **transient** motion y_{tr} and **steady periodic** oscillations y_{sp}.

Example 111. Find the steady periodic solution to $y'' + 2y' + 5y = \cos(\lambda t)$. What is the amplitude of the steady periodic oscillations? For which ω is the amplitude maximal?

Solution. The "old" roots are $-1 \pm 2i$.

[Not really needed, because positive damping prevents duplication; can you see it?]

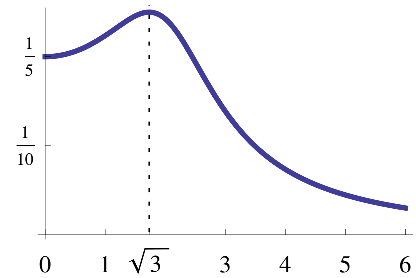
Hence, $y_{\text{sp}} = A\cos(\lambda t) + B\sin(\lambda t)$ and to find A, B we need to plug into the DE.

Doing so, we find $A = \frac{5-\lambda^2}{(5-\lambda^2)^2 + 4\lambda^2}$, $B = \frac{2\lambda}{(5-\lambda^2)^2 + 4\lambda^2}$.

Thus, the amplitude of y_{sp} is $r(\lambda) = \sqrt{A^2 + B^2} = \frac{1}{\sqrt{(5-\lambda^2)^2 + 4\lambda^2}}$.

The function $r(\lambda)$ is sketched to the right. It has a maximum at $\lambda = \sqrt{3}$ at which the amplitude is unusually large (well, here it is not very pronounced). We say that **practical resonance** occurs for $\lambda = \sqrt{3}$.

[For comparison, without damping, (pure) resonance occurs for $\lambda = \sqrt{5}$.]



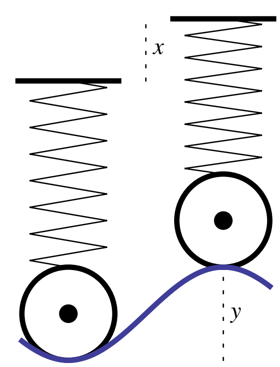
Example 112. A car is going at constant speed v on a washboard road surface ("bumpy road") with height profile $y(s) = a\cos\left(\frac{2\pi s}{L}\right)$. Assume that the car oscillates vertically as if on a spring (no dashpot). Describe the resulting oscillations.

Solution. With x as in the sketch, the spring is stretched by $x - y$. Hence, by Hooke's and Newton's laws, its motion is described by $mx'' = -k(x-y)$.

At constant speed, $s = vt$ and we obtain the DE $mx'' + kx = ky = ka\cos\left(\frac{2\pi vt}{L}\right)$, which is inhomogeneous linear with constant coefficients. Let's solve it.

"Old" roots: $\pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$. $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency.

"New" roots: $i\frac{2\pi v}{L} = \pm i\omega$. $\omega = \frac{2\pi v}{L}$ is the external frequency.



The first "car" is assumed to be in equilibrium.

Case 1: $\omega \neq \omega_0$. Then a particular solution is $x_p = b_1\cos(\omega t) + b_2\sin(\omega t) = A\cos(\omega t - \alpha)$ for unique values of b_1, b_2 (which we do not compute here). The general solution is of the form $x = x_p + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$.

Case 2: $\omega = \omega_0$. Then a particular solution is $x_p = t[b_1\cos(\omega t) + b_2\sin(\omega t)] = At\cos(\omega t - \alpha)$ for unique values of b_1, b_2 (which we do not compute). Note that the amplitude in x_p increases without bound; the same is true for the general solution $x = x_p + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$. This phenomenon is called resonance; it occurs if an external frequency matches a natural frequency.

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The Laplace transform

Definition 113. The **Laplace transform** of a function $f(t)$, $t \geqslant 0$, is defined as the new function

$$F(s) = \int_0^\infty e^{-st} f(t) \mathrm{d}t.$$

We also write $\mathcal{L}(f(t)) = F(s)$.

Note that, in order for the integral to exist, $f(t)$ should be, say, piecewise continuous and of at most exponential growth. That's true for most of the functions, we are interested in (and so we will not dwell on this issue).

$f(t)$	$F(s)$
e^{at}	$\frac{1}{s-a}$
1	$\frac{1}{s}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - s f(0) - f'(0)$

First entries in the Laplace transform table

In this section, we will discuss and obtain the entries in the table of the most basic Laplace transforms that we compiled after Definition 113.

Example 114. Show that $\mathcal{L}(e^{at}) = \frac{1}{s-a}$.

In particular, in the special case $a = 0$, we have $\mathcal{L}(1) = \frac{1}{s}$.

Solution. $\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at} \mathrm{d}t = \int_0^\infty e^{(a-s)t} \mathrm{d}t = \left[\frac{1}{a-s} e^{(a-s)t}\right]_{t=0}^\infty = 0 - \frac{1}{a-s} = \frac{1}{s-a}$

Comment. Note that we needed $a - s < 0$ in order for the integral to converge. Hence the Laplace transform has domain $s > a$. (During this introduction, we will not care too much about these technical details.)

In particular. Note that setting $a = 0$ shows that $\mathcal{L}(1) = \frac{1}{s}$.

Example 115. (linearity) Show that $\mathcal{L}(c_1 f_1(t) + c_2 f_2(t)) = c_1 F_1(s) + c_2 F_2(s)$.

This means that the Laplace transform is a **linear operator** (like the derivative or the integral).

Solution.

$$\begin{aligned}\mathcal{L}(c_1 f_1(t) + c_2 f_2(t)) &= \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t)) \mathrm{d}t \\ &= c_1 \underbrace{\int_0^\infty e^{-st} f_1(t) \mathrm{d}t}_{F_1(s)} + c_2 \underbrace{\int_0^\infty e^{-st} f_2(t) \mathrm{d}t}_{F_2(x)}\end{aligned}$$

Example 116. (extra) Show that $\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$ and $\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2}$.

Solution. By Euler's identity, $e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$. Hence, by linearity,

$$\mathcal{L}(e^{i\omega t}) = \mathcal{L}(\cos(\omega t)) + i \mathcal{L}(\sin(\omega t)).$$

On the other hand,

$$\mathcal{L}(e^{i\omega t}) = \frac{1}{s - i\omega} = \frac{s + i\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} + i \frac{\omega}{s^2+\omega^2}.$$

Matching real and imaginary parts, we find $\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$ and $\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2}$.

Example 117. Determine $\mathcal{L}(e^{3t} - 7e^{-2t})$.

Solution. $\mathcal{L}(e^{3t} - 7e^{-2t}) = \mathcal{L}(e^{3t}) - 7\mathcal{L}(e^{-2t}) = \frac{1}{s-3} - \frac{7}{s+2}$

Comment. Note that, once we write $\frac{1}{s-3} - \frac{7}{s+2} = -\frac{6s-23}{s^2-s-6}$ it is no longer visibly clear which function we have taken the Laplace transform of. We discuss reversing this process in the next section.

Example 118. (extra) Determine $\mathcal{L}(3\cos(2t) - 5\sin(2t))$.

Solution. $\mathcal{L}(3\cos(2t) - 5\sin(2t)) = 3\mathcal{L}(\cos(2t)) - 5\mathcal{L}(\sin(2t)) = 3\frac{s}{s^2+4} - 5\frac{2}{s^2+4} = \frac{3s-10}{s^2+4}$

Example 119. Show that $\mathcal{L}(f'(t)) = sF(s) - f(0)$.

Solution. Using integration by parts,

$$\mathcal{L}(f'(t)) = \int_0^\infty e^{-st} f'(t) \mathrm{d}t = \left[e^{-st} f(t)\right]_{t=0}^\infty + \int_0^\infty s e^{-st} f(t) \mathrm{d}t = sF(s) - f(0).$$

Higher derivatives. In order to obtain the Laplace transform of higher derivatives, we can iterate. For instance,

$$\mathcal{L}(f''(t)) = s\mathcal{L}(f'(t)) - f'(0) = s[sF(s) - f(0)] - f'(0) = s^2F(s) - sf(0) - f'(0).$$

The inverse Laplace transform

Theorem 120. (uniqueness of Laplace transforms) If $\mathcal{L}(f_1(t)) = \mathcal{L}(f_2(t))$, then $f_1(t) = f_2(t)$. Hence, we can recover $f(t)$ from $F(s)$. We write $\mathcal{L}^{-1}(F(s)) = f(t)$.

We say that $f(t)$ is the **inverse Laplace transform** of $F(s)$.

Advanced comment. This uniqueness is true for continuous functions f_1, f_2. It is also true for piecewise continuous functions except at those values of t for which there is a discontinuity. (Note that redefining $f(t)$ at a single point, will not change its Laplace transform.)

Example 121. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{5}{s+3}\right)$.

Solution. In other words, if $F(s) = \frac{5}{s+3}$, what is $f(t)$?

$\mathcal{L}^{-1}\left(\frac{5}{s+3}\right) = 5\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = 5e^{-3t}$

Example 122. (extra) Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{3s-7}{s^2+4}\right)$.

Solution. In other words, if $F(s) = \frac{3s-7}{s^2+4}$, what is $f(t)$?

$F(s) = 3\frac{s}{s^2+2^2} - \frac{7}{2}\frac{2}{s^2+2^2}$. Hence, $f(t) = 3\cos(2t) - \frac{7}{2}\sin(2t)$.

Example 123. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right)$.

Solution. Note that $s^2 - s - 6 = (s-3)(s+2)$. We use **partial fractions** to write $-\frac{6s-23}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$. We find the coefficients as

$$A = -\frac{6s-23}{s+2}\bigg|_{s=3} = 1, \quad B = -\frac{6s-23}{s-3}\bigg|_{s=-2} = -7.$$

Hence $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-3} - \frac{7}{s+2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) - 7\mathcal{L}^{-1}\left(\frac{7}{s+2}\right) = e^{3t} - 7e^{-2t}$.

Review. In order to find A, we multiply $-\frac{6s-23}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$ by $s-3$ to get $-\frac{6s-23}{s+2} = A + \frac{B(s-3)}{s+2}$. We then set $s = 3$ to find A as above.

Comment. Compare with Example 117 where we considered the same functions.

Example 124. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{s+13}{s^2-s-2}\right)$.

Solution. Note that $s^2-s-2=(s-2)(s+1)$. We use partial fractions to write $\frac{s+13}{(s-2)(s+1)}=\frac{A}{s-2}+\frac{B}{s+1}$. We find the coefficients as

$$A=\left.\frac{s+13}{s+1}\right|_{s=2}=5, \quad B=\left.\frac{s+13}{s-2}\right|_{s=-1}=-4.$$

Hence $\mathcal{L}^{-1}\left(\frac{s+13}{s^2-s-2}\right)=\mathcal{L}^{-1}\left(\frac{5}{s+1}-\frac{4}{s-2}\right)=5e^t-4e^{2t}$.

Solving simple DEs using the Laplace transform

In the following examples, we write $Y(s)$ for the Laplace transform of $y(t)$.

Example 125. Solve the (very simple) IVP $y'(t)-2y(t)=0$, $y(0)=7$.

At this point, you might be able to "see" right away that the unique solution is $y(t)=7e^{2t}$.

Solution. (old style) The characteristic root is 2, so that the general solution is $y(t)=Ce^{2t}$. Using the initial condition, we find that $C=7$, so that $y(t)=7e^{2t}$.

Solution. (Laplace style) $y' - 2y = 0$ transforms into

$$\mathcal{L}(y'(t)-2y(t))=\mathcal{L}(y'(t))-2\mathcal{L}(y(t))= sY(s)-y(0) -2Y(s) =(s-2)Y(s)-7=0.$$

This is an algebraic equation for $Y(s)$. It follows that $Y(s)=\frac{7}{s-2}$. By inverting the Laplace transform, we conclude that $y(t)=7e^{2t}$.

Example 126. Solve the IVP $y'' - 3y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$.

Solution. (old style) The characteristic polynomial $D^2 - 3D + 2 = (D-1)(D-2)$ has ("old") roots $1, 2$.
The "new" root is -1. Since there is no duplication, there must be a particular solution of the form $y_p(t) = Ae^{-t}$.
To determine A, we plug into the DE $y_p'' - 3y_p' + 2y_p = 6Ae^{-t} \overset{!}{=} e^{-t}$ and conclude $A = \frac{1}{6}$.
The general solution thus is $y(t) = \frac{1}{6}e^{-t} + C_1 e^t + C_2 e^{2t}$. We need to find C_1 and C_2 using the initial conditions.
Solving $y(0) = \frac{1}{6} + C_1 + C_2 \overset{!}{=} 0$ and $y'(0) = -\frac{1}{6} + C_1 + 2C_2 \overset{!}{=} 1$, we find $C_2 = \frac{4}{3}$ and $C_1 = -\frac{3}{2}$.
Hence, the unique solution to the IVP is $y(t) = \frac{1}{6}e^{-t} - \frac{3}{2}e^t + \frac{4}{3}e^{2t}$.

Solution. (Laplace style) The differential equation transforms as follows:

$$\begin{aligned}
\mathcal{L}(y''(t)) - 3\mathcal{L}(y'(t)) + 2\mathcal{L}(y(t)) &= \mathcal{L}(e^{-t}) \\
s^2 Y(s) - s y(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) &= \frac{1}{s+1} \\
(s^2 - 3s + 2)Y(s) &= 1 + \frac{1}{s+1} = \frac{s+2}{s+1} \\
Y(s) &= \frac{s+2}{(s-1)(s-2)(s+1)}
\end{aligned}$$

To find $y(t)$, we use partial fractions to write $Y(s) = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+1}$. We find the coefficients as

$$A = \frac{s+2}{(s-2)(s+1)}\bigg|_{s=1} = -\frac{3}{2}, \quad B = \frac{s+2}{(s-1)(s+1)}\bigg|_{s=2} = \frac{4}{3}, \quad C = \frac{s+2}{(s-1)(s-2)}\bigg|_{s=-1} = \frac{1}{6}.$$

Hence, $y(t) = \mathcal{L}^{-1}\left(\frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+1}\right) = Ae^t + Be^{2t} + Ce^{-t} = \frac{1}{6}e^{-t} - \frac{3}{2}e^t + \frac{4}{3}e^{2t}$, as above.

Comment. Note the factor $s^2 - 3s + 2$ in front of $Y(s)$ when we transformed the DE. This is the characteristic polynomial. Can you see how the "old" and "new" roots show up in the Laplace transform approach?

Example 127. Consider the IVP $y'' - 3y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$.
Determine the Laplace transform of the unique solution.

Solution. We just did that! By transforming the DE, we found that $Y(s) = \frac{s+2}{(s-1)(s-2)(s+1)}$.

Example 128. Consider the IVP $y'' - 3y' + y = 2e^{5t}$, $y(0) = -1$, $y'(0) = 4$.
Determine the Laplace transform of the unique solution.

Solution. The DE $y'' - 3y' + y = 2e^{5t}$ transforms into

$$s^2 Y - s y(0) - y'(0) - 3(sY - y(0)) + Y = (s^2 - 3s + 1)Y + (s - 7) = \frac{2}{s-5}.$$

Accordingly, $Y(s) = \frac{1}{s^2 - 3s + 1}\left[\frac{2}{s-5} - s + 7\right]$ is the Laplace transform of the unique solution to the IVP.

Comment. The characteristic roots are $(3 \pm \sqrt{5})/2$. As a result, the solution $y(t)$ will be rather unpleasant to write down by hand, with coefficients that are not rational numbers. By contrast, the above Laplace transform can be expressed without irrational numbers.

Comment. Depending on what we intend to do with the solution, we might not even need $y(t)$ but might instead be able to extract what we want from its Laplace transform $Y(s)$.

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We solved the following system in Example 92 using elimination and our method for solving linear DEs with constant coefficients based on characteristic roots.

Example 129. (extra) Solve the system $y_1' = 5y_1 + 4y_2$, $y_2' = 8y_1 + y_2$, $y_1(0) = 0$, $y_2(0) = 1$.

Solution. (using Laplace transforms) $y_1' = 5y_1 + 4y_2$ transforms into $sY_1 - \underbrace{y_1(0)}_{=0} = 5Y_1 + 4Y_2$.

Likewise, $y_2' = 8y_1 + y_2$ transforms into $sY_2 - \underbrace{y_2(0)}_{=1} = 8Y_1 + Y_2$.

The transformed equations are regular equations that we can solve for Y_1 and Y_2.

For instance, by the first equation, $Y_2 = \frac{1}{4}(s-5)Y_1$.

Used in the second equation, we get $\underbrace{-8Y_1 + \frac{1}{4}(s-1)(s-5)Y_1}_{=\frac{1}{4}(s^2-6s-27)=\frac{1}{4}(s+3)(s-9)} = 1$ so that $Y_1 = \frac{4}{(s+3)(s-9)}$.

Hence, the system is solved by $Y_1 = \frac{4}{(s+3)(s-9)}$ and $Y_2 = \frac{1}{4}(s-5)Y_1 = \frac{s-5}{(s+3)(s-9)}$.

As a final step, we need to take the inverse Laplace transform to get $y_1(t) = \mathcal{L}^{-1}(Y_1(s))$ and $y_2(t) = \mathcal{L}^{-1}(Y_2(s))$.

Using partial fractions, $Y_1(s) = \frac{4}{(s+3)(s-9)} = -\frac{1}{3} \cdot \frac{1}{s+3} + \frac{1}{3} \cdot \frac{1}{s-9}$ so that $y_1(t) = -\frac{1}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

Similarly, $Y_2(s) = \frac{s-5}{(s+3)(s-9)} = \frac{2}{3} \cdot \frac{1}{s+3} + \frac{1}{3} \cdot \frac{1}{s-9}$ so that $y_1(t) = \frac{2}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

Solution. (old solution, for comparison) Since $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1$ (from the first eq.), we have $y_2' = \frac{1}{4}y_1'' - \frac{5}{4}y_1'$.

Using these in the second equation, we get $\frac{1}{4}y_1'' - \frac{5}{4}y_1' = 8y_1 + \frac{1}{4}y_1' - \frac{5}{4}y_1$.
Simplified, this is $y_1'' - 6y_1' - 27y_1 = 0$.
This is a homogeneous linear DE with constant coefficients. The characteristic roots are $-3, 9$.
We therefore obtain $y_1 = C_1e^{-3t} + C_2e^{9t}$ as the general solution.
Thus, $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1 = \frac{1}{4}(-3C_1e^{-3t} + 9C_2e^{9t}) - \frac{5}{4}(C_1e^{-3t} + C_2e^{9t}) = -2C_1e^{-3t} + C_2e^{9t}$.
We determine the (unique) values of C_1 and C_2 using the initial conditions:
$y_1(0) = C_1 + C_2 \overset{!}{=} 0$
$y_2(0) = -2C_1 + C_2 \overset{!}{=} 1$
We solve these two equations and find $C_1 = -\frac{1}{3}$ and $C_2 = \frac{1}{3}$.
The unique solution to the IVP therefore is $y_1(t) = -\frac{1}{3}e^{-3t} + \frac{1}{3}e^{9t}$ and $y_2(t) = \frac{2}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

Further entries in the Laplace transform table

We next expand our table of Laplace transforms to the following:

$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - s\,f(0) - f'(0)$
e^{at}	$\frac{1}{s-a}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{at}f(t)$	$F(s-a)$
$t\,f(t)$	$-F'(s)$
$u_a(t)\,f(t-a)$	$e^{-sa}F(s)$

Example 130.

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-st}e^{at}f(t)\mathrm{d}t = \int_0^\infty e^{-(s-a)t}f(t)\mathrm{d}t = F(s-a)$$

Example 131. We also add the following to our table of Laplace transforms.

$$\mathcal{L}(tf(t)) = \int_0^\infty e^{-st}tf(t)\mathrm{d}t = \int_0^\infty -\frac{\mathrm{d}}{\mathrm{d}s}e^{-st}f(t)\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st}f(t)\mathrm{d}t = -F'(s)$$

In particular,

$$\begin{aligned} \mathcal{L}(t) &= \mathcal{L}(t\cdot 1) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s} = \frac{1}{s^2} \\ \mathcal{L}(t^2) &= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s^2} = \frac{2}{s^3} \\ &\vdots \\ \mathcal{L}(t^n) &= \frac{n!}{s^{n+1}}. \end{aligned}$$

Example 132. Determine the Laplace transform $\mathcal{L}((t-3)e^{2t})$.

Solution. $\mathcal{L}((t-3)e^{2t}) = \mathcal{L}(te^{2t}) - 3\mathcal{L}(e^{2t}) = \frac{1}{(s-2)^2} - \frac{3}{s-2}$

Here, we combined $\mathcal{L}(tf(t)) = -F'(s)$ with $\mathcal{L}(e^{2t}) = \frac{1}{s-2}$ to get $\mathcal{L}(te^{2t}) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s-2} = \frac{1}{(s-2)^2}$.

Example 133. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right)$.

Solution. $\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right) = e^{3t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = te^{3t}$.

Example 134. (bonus) Solve the IVP $y'' - 3y' + 2y = e^t$, $y(0) = 0$, $y'(0) = 1$.

Solution. (old style, outline) The characteristic polynomial $D^2 - 3D + 2 = (D-1)(D-2)$. Since there is duplication, we have to look for a particular solution of the form $y_p = Ate^t$. To determine A, we need to plug into the DE (we find $A = -1$). Then, the general solution is $y(t) = Ate^t + C_1e^t + C_2e^{2t}$, and the initial conditions determine C_1 and C_2 (we find $C_1 = -2$ and $C_2 = 2$).

Solution. (Laplace style)

$$\begin{aligned} \mathcal{L}(y''(t)) - 3\mathcal{L}(y'(t)) + 2\mathcal{L}(y(t)) &= \mathcal{L}(e^t) \\ s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) &= \frac{1}{s-1} \\ (s^2 - 3s + 2)Y(s) &= 1 + \frac{1}{s-1} = \frac{s}{s-1} \\ Y(s) &= \frac{s}{(s-1)^2(s-2)} \end{aligned}$$

To find $y(t)$, we again use partial fractions. $Y(s) = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-2}$ with coefficients (why?!)

$$C = \left.\frac{s}{(s-1)^2}\right|_{s=2} = 2, \quad A = \left.\frac{s}{s-2}\right|_{s=1} = -1, \quad B = \left.\frac{\mathrm{d}}{\mathrm{d}s}\frac{s}{s-2}\right|_{s=1} = \left.\frac{-2}{(s-2)^2}\right|_{s=1} = -2.$$

Finally, $y(t) = \mathcal{L}^{-1}\left(\frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-2}\right) = Ate^t + Be^t + Ce^{2t} = -(t+2)e^t + 2e^{2t}$.

Let $u_a(t) = \begin{cases} 1, & \text{if } t \geqslant a, \\ 0, & \text{if } t < a, \end{cases}$ be the **unit step function**.

Comment. The special case $u_0(t)$ is also known as the **Heaviside function**, after Oliver Heaviside who, among many other things, coined terms like conductance and impedance. Note that $u_a(t) = u_0(t-a)$.

Example 135. $$\mathcal{L}(u_a(t)) = \int_0^\infty e^{-st} u_a(t) \mathrm{d}t = \int_a^\infty e^{-st} \mathrm{d}t = \left[-\frac{e^{-st}}{s} \right]_{t=a}^\infty = \frac{e^{-sa}}{s}.$$

Example 136. Note that $u_a(t) f(t-a)$ is $f(t)$ delayed by a (make a sketch!). We find

$$\mathcal{L}(u_a(t) f(t-a)) = \int_a^\infty e^{-st} f(t-a) \mathrm{d}t = \int_0^\infty e^{-s(\tilde{t}+a)} f(\tilde{t}) \mathrm{d}\tilde{t} = e^{-sa} F(s).$$

Example 137. What is $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+1} \right)$?

Solution. $\frac{1}{s+1}$ is the Laplace transform of e^{-t}. Hence, $\frac{e^{-2s}}{s+1}$ is the Laplace transform of e^{-t} delayed by 2. In other words, $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+1} \right) = u_2(t) e^{-(t-2)}$.

The next example illustrates that any piecewise defined function can be written using a single formula involving step functions. This is based on the simple observation that $u_a(t) - u_b(t)$ is a function which is 1 on the interval $[a, b)$ but zero everywhere else.

Example 138. Consider $f(t) = \begin{cases} t^2, & \text{if } 0 \leqslant t \leqslant 1, \\ 1, & \text{if } 1 \leqslant t \leqslant 2, \\ \cos(t-2), & \text{if } t \geqslant 2. \end{cases}$

Then, $f(t) = t^2 (u_0(t) - u_1(t)) + 1(u_1(t) - u_2(t)) + \cos(t-2) u_2(t)$.

It is left as an exercise to compute the Laplace transform of $f(t)$ from here. Note that, for instance, to find $\mathcal{L}(t^2 u_1(t))$, we want to use $\mathcal{L}(u_a(t) f(t-a)) = e^{-sa} F(s)$ with $a = 1$ and $f(t-1) = t^2$; then, $f(t) = (t+1)^2 = t^2 + 2t + 1$ has Laplace transform $F(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$, and we combine to get $\mathcal{L}(t^2 u_1(t)) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \right)$.

Example 139. Solve the IVP $y'' + 3y' + 2y = f(t)$, $y(0) = y'(0) = 0$ with $f(t) = \begin{cases} 1, & t \in [3,4], \\ 0, & \text{otherwise.} \end{cases}$

Solution. First, we write $f(t) = u_3(t) - u_4(t)$. We can now take the Laplace transform of the DE to get

$$s^2 Y(s) - s y(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) = \frac{e^{-3s}}{s} - \frac{e^{-4s}}{s} = (e^{-3s} - e^{-4s})\frac{1}{s}.$$

Using that $s^2 + 3s + 2 = (s+1)(s+2)$, we find

$$Y(s) = (e^{-3s} - e^{-4s})\frac{1}{s(s+1)(s+2)} = (e^{-3s} - e^{-4s})\left[\frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}\right],$$

where A, B, C are determined by partial fractions (we compute them below). Taking the inverse Laplace transform of each of the six terms in this product, as in Example 137, we find

$$y(t) = A(u_3(t) - u_4(t)) + B(u_3(t)e^{-(t-3)} - u_4(t)e^{-(t-4)}) + C(u_3(t)e^{-2(t-3)} - u_4(t)e^{-2(t-4)}).$$

If preferred, we can express this as $y(t) = \begin{cases} 0, & \text{if } t \leqslant 3, \\ A + Be^{-(t-3)} + Ce^{-2(t-3)}, & \text{if } t \in [3,4], \\ B(e^{-(t-3)} - e^{-(t-4)}) + C(e^{-2(t-3)} - e^{-2(t-4)}) & \text{if } t \geqslant 4. \end{cases}$

Finally, $A = \frac{1}{(s+1)(s+2)}\Big|_{s=0} = \frac{1}{2}$, $B = \frac{1}{s(s+2)}\Big|_{s=-1} = -1$, $C = \frac{1}{s(s+1)}\Big|_{s=-2} = \frac{1}{2}$.

Comment. Check that these values make $y(t)$ a continuous function (as it should be for physical reasons).