Review: Computing derivatives

Given a function $y(x)$, we learned in Calculus I that its **derivative**

$$y'(x) = \frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

(where $\Delta y = y(x + \Delta x) - y(x)$) has the following two important characterizations:

- $y'(x)$ is the **slope of the tangent line** of the graph of $y(x)$ at x, and
- $y'(x)$ is the **rate of change** of $y(x)$ at x.

Moreover, we learned simple rules to compute the derivative of functions:

- **(sum rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)] = f'(x) + g'(x)$
- **(product rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
- **(chain rule)** $\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x))] = f'(g(x))g'(x)$

  Comment. If we write $t = g(x)$ and $y = f(t)$, then the chain rule takes the form $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x}$. In other words, the chain rule expresses the fact that we can treat $\frac{\mathrm{d}y}{\mathrm{d}x}$ (which initially is just a notation for $y'(x)$) as an honest fraction.

- **(basic functions)** $\frac{\mathrm{d}}{\mathrm{d}x} x^r = r x^{r-1}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$, $\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = \frac{1}{x}$,
  $\frac{\mathrm{d}}{\mathrm{d}x} \sin(x) = \cos(x)$, $\frac{\mathrm{d}}{\mathrm{d}x} \cos(x) = -\sin(x)$

These rules are enough to compute the derivative of any function that we can build from the basic functions using algebraic operations and composition. On the other hand, as you probably recall from Calculus II, reversing the operation of differentiation (i.e. computing antiderivatives) is much more difficult.

In particular, there exist simple functions (such as e^{x^2}) whose antiderivative cannot be expressed in terms of the basic functions above.

Example 1. Derive the **quotient rule** from the rules above.

Solution. We write $\frac{f(x)}{g(x)} = f(x) \cdot \frac{1}{g(x)}$ and apply the product rule to get

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x) \frac{1}{g(x)} + f(x) \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{g(x)}.$$

By the chain rule combined with $\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{x} = -\frac{1}{x^2}$, we have $\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{g(x)} = -\frac{1}{g(x)^2} g'(x)$. Using this in the previous formula,

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \cdot \frac{1}{g(x)} = f'(x) \frac{1}{g(x)} - f(x) \frac{1}{g(x)^2} g'(x) = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{g(x)^2}.$$

Putting the final two fractions on a common denominator, we obtain the familiar quotient rule

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

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First examples of differential equations

Example 2. If $y(x)=e^{x^2}$ then $y'(x)=2xe^{x^2}=2xy(x)$ or, for short, $y'=2xy$.

Accordingly, we say that $y(x)=e^{x^2}$ is a **solution** to the **differential equation** (DE) $y'=2xy$.

Example 3. By computing its derivative, determine a DE solved by $y(x)=\sin(3x)$.

Solution. $y'(x)=3\cos(3x)=3\sqrt{1-(\sin(3x))^2}=3\sqrt{1-y(x)^2}$

[Here we used that $\cos(x)^2+\sin(x)^2=1$, which implies that $\cos(x)=\sqrt{1-\sin(x)^2}$.]

Hence, $y(x)=\sin(3x)$ solves the differential equation $y'=3\sqrt{1-y^2}$.

Example 4. By computing its second derivative, determine another DE solved by $y(x)=\sin(3x)$.

Solution. $y''(x)=-9\sin(3x)=-9y(x)$.
Thus, $y(x)=\sin(3x)$ also solves the differential equation $y''=-9y$.

If the highest derivative appearing in a DE is an rth derivative, we say that the DE has **order** r.

For instance. The DE $y'=3\sqrt{1-y^2}$ has order 1 (such DEs are also called first order DEs).
On the other hand, the DE $y''=-9y$ has order 2 (such DEs are also called second order DEs).

Example 5. Verify that $e^y y' = 1$ is solved by $y(x) = \ln(x + C)$.

Solution. $y'(x) = \frac{1}{x + C}$ and $e^{y(x)} = x + C$.

Hence, $e^y y' = (x + C)\frac{1}{x + C} = 1$.

Because $y(x)$ solves the DE for any value of the parameter C, we say that $y(x) = \ln(x + C)$ is a **one-parameter family** of solutions to the DE.

Example 6. Consider the DE $y'' = y' + 6y$. For which r is e^{rx} a solution?

Solution. If $y(x) = e^{rx}$, then $y'(x) = r e^{rx}$ and $y''(x) = r^2 e^{rx}$.

Plugging $y(x) = e^{rx}$ into the DE, we get $r^2 e^{rx} = r e^{rx} + 6e^{rx}$ which simplifies to $r^2 = r + 6$.

This has the two solutions $r = -2$, $r = 3$. Hence e^{-2x} and e^{3x} are solutions of the DE.

In fact, we check that $Ae^{-2x} + Be^{3x}$ is a **two-parameter family** of solutions to the DE.

Important comment. It is no coincidence that the order of the DE is 2, whereas the previous example has order 1. In general, we expect a DE of order r to have a solution with r parameters.

Example 7. Solve the DE $y' = x^2 + x$.

Solution. Note that the DE simply asks for a function $y(x)$ with a specific derivative (in particular, the right-hand side does not involve $y(x)$). In other words, the desired $y(x)$ is an **antiderivative** of $x^2 + x$. We know from Calculus II that we can find antiderivatives by integrating:

$$y(x) = \int (x^2 + x)\mathrm{d}x = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$$

Moreover, we know from Calculus II that there are no other solutions. In other words, we found the **general solution** to the DE.

To single out a **particular solution**, we need to specify additional conditions (typically one condition per parameter in the general solution). For instance, it is common to impose **initial conditions** such as $y(1) = 2$. A DE together with an initial condition is called an **initial value problem** (IVP).

Example 8. Solve the IVP $y' = x^2 + x$ with $y(1) = 2$.

Solution. From the previous example, we know that $y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + C$.

Since $y(1) = \frac{1}{3} + \frac{1}{2} + C = \frac{5}{6} + C \overset{!}{=} 2$, we find $C = 2 - \frac{5}{6} = \frac{7}{6}$.

Hence, $y(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{7}{6}$ is the (unique) solution of the IVP.

Example 9. Solve the DE $y'' = x^2 + x$.

Solution. We now take two antiderivatives of $x^2 + x$ to get

$$y(x) = \iint (x^2 + x)\mathrm{d}x\mathrm{d}x = \int \left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + C\right)\mathrm{d}x = \frac{1}{12}x^4 + \frac{1}{6}x^3 + Cx + D,$$

where it is important that we give the second constant of integration a name different from the first.

Again, this is the **general solution** to the DE. The DE is of order 2 and, as expected, the general solution has 2 parameters.

Important. Note that we are working with functions $y(x)$ of a single variable. This allows us to write simply y' for $\frac{\mathrm{d}}{\mathrm{d}x}y(x)$ without risk of confusion.

Of course, we may use different variables such as $x(t)$ and $x' = \frac{\mathrm{d}}{\mathrm{d}t}x(t)$, as long as this is clear from the context.

Differential equations that involve only derivatives with respect to a single variable are known as **ordinary differential equations** (ODEs).

On the other hand, differential equations that involve derivatives with respect to several variables are referred to as **partial differential equations** (PDEs).

Example 10. The DE

$$\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^2 u(x,y) + \left(\frac{\mathrm{d}}{\mathrm{d}y}\right)^2 u(x,y) = 0,$$

often abbreviated as $u_{xx} + u_{yy} = 0$, is a partial differential equation in two variables.

This particular PDE is known as **Laplace's equation** and describes, for instance, steady-state heat distributions.
https://en.wikipedia.org/wiki/Laplace%27s_equation

This and other fundamental PDEs will be discussed in Differential Equations II.

Example 11. (review)

(a) Verify that $x(t) = \frac{1}{c - kt}$ is a one-parameter family of solutions to the DE $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$.

(b) Solve the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$, $x(0) = 2$.

(c) Solve the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$, $x(0) = 0$.

Solution.

(a) We compute that $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{(c - kt)^2} \cdot (-k) = \frac{k}{(c - kt)^2}$.

On the other hand, $kx^2 = k\left(\frac{1}{c - kt}\right)^2 = \frac{k}{(c - kt)^2}$ as well. Thus, indeed, $\frac{\mathrm{d}x}{\mathrm{d}t} = kx^2$.

(b) We start with $x(t) = \frac{1}{c - kt}$ (which we know solves the DE for any value of c) and seek to choose c so that $x(0) = 2$.

Since $x(0) = \left[\frac{1}{c - kt}\right]_{t=0} = \frac{1}{c} \stackrel{!}{=} 2$, we find $c = \frac{1}{2}$.

Hence, the IVP has the (unique) solution $x(t) = \frac{1}{1/2 - kt}$.

(c) Proceeding as in the previous part, we now arrive at the impossible equation $\frac{1}{c} \stackrel{!}{=} 0$.

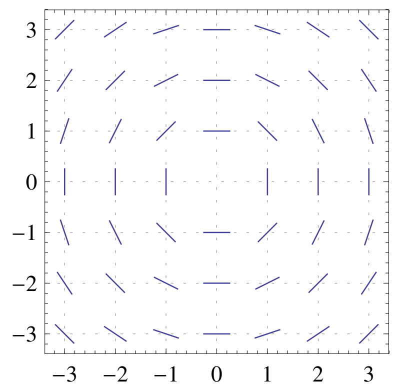
However, this suggests that we should consider taking $c \to \infty$ in $x(t) = \frac{1}{c - kt}$, which results in $x(t) = 0$.

Indeed, it is easy to verify (make sure you know what this entails!) that $x(t) = 0$ solves the IVP.

Slope fields, or sketching solutions to DEs

Example 12. Consider the DE $y' = -x/y$.

Let's pick a point, say, $(1,2)$. If a solution $y(x)$ is passing through that point, then its slope has to be $y' = -1/2$. We therefore draw a small line through the point $(1,2)$ with slope $-1/2$. Continuing in this fashion for several other points, we obtain the **slope field** on the right.



With just a little bit of imagination, we can now anticipate the solutions to look like (half)circles around the origin. Let us check whether $y(x) = \sqrt{r^2 - x^2}$ might indeed be a solution!

$y'(x) = \frac{1}{2}\frac{-2x}{\sqrt{r^2 - x^2}} = -x/y(x)$. So, yes, we actually found solutions!

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Solving DEs: Separation of variables

Example 13. Solve the DE $y' = -\frac{x}{y}$.

Solution. Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$.

Separate the variables to get $y\,\mathrm{d}y = -x\,\mathrm{d}x$ (in particular, we are multiplying both sides by $\mathrm{d}x$).

Integrating both sides, we get $\int y\,\mathrm{d}y = \int -x\,\mathrm{d}x$.

Computing both integrals results in $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$ (we combine the two constants of integration into one).

Hence $x^2 + y^2 = D$ (with $D = 2C$).

This is an **implicit form** of the solutions to the DE. We can make it explicit by solving for y. Doing so, we find $y(x) = \pm\sqrt{D - x^2}$ (choosing $+$ gives us the upper half of a circle, while the negative sign gives us the lower half).

Comment. The step above where we break $\frac{\mathrm{d}y}{\mathrm{d}x}$ apart and then integrate may sound sketchy!

However, keep in mind that, after we find a solution $y(x)$, even if by sketchy means, we can (and should!) verify that $y(x)$ is indeed a solution by plugging into the DE. We actually already did that in the previous example!

In general, **separation of variables** solves $y' = g(x)h(y)$ by writing the DE as $\frac{1}{h(y)}\mathrm{d}y = g(x)\mathrm{d}x$.

Note that $\frac{1}{h(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$ is indeed equivalent to $\int \frac{1}{h(y)}\mathrm{d}y = \int g(x)\mathrm{d}x + C$. Why?! (Apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to the integrals...)

Example 14. Solve the IVP $y' = -\frac{x}{y}$, $y(0) = -3$.

Comment. Instead of using what we found in the previous example, we start from scratch to better illustrate the solution process (and how we can use the initial condition right away to determine the value of the constant of integration).

Solution. We separate variables to get $y\,\mathrm{d}y = -x\,\mathrm{d}x$.

Integrating gives $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$, and we use $y(0) = -3$ to find $\frac{1}{2}(-3)^2 = 0 + C$ so that $C = \frac{9}{2}$.

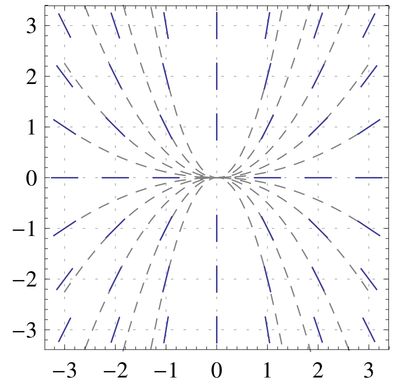
Hence, $x^2 + y^2 = 9$ is an **implicit** form of the solution.

Solving for y, we get $y = -\sqrt{9 - x^2}$ (note that we have to choose the negative sign so that $y(0) = -3$).

Comment. Note that our solution is a **local solution**, meaning that it is valid (and solves the DE) locally around $x = 0$ (from the initial condition). However, it is not a **global solution** because it doesn't make sense outside of x in the interval $[-3, 3]$.

Example 15. Consider the DE $xy' = 2y$.

Sketch its slope field.



Challenge. Try to guess solutions $y(x)$ from the slope field.

Solution. For instance, to find the slope at the point $(3, 1)$, we plug $x = 3$, $y = 1$ into the DE to get $3y' = 2$. Hence, the slope is $y' = 2/3$.

The resulting slope field is sketched on the right.

Solution of the challenge. Trace out the solution through $(1, 1)$ (and then some other points). Their shape looks like a parabola, so that we might guess that $y(x) = Cx^2$ solves the DE. Check that this is indeed the case by plugging into the DE!

Example 16. Solve the IVP $xy' = 2y$, $y(1) = 2$.

Solution. Rewrite the DE as $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x}$.

Then multiply both sides with $\mathrm{d}x$ and integrate both of them to get $\int \frac{1}{y}\mathrm{d}y = \int \frac{2}{x}\mathrm{d}x$.

Hence, $\ln|y| = 2\ln|x| + C$.

The initial condition $y(1) = 2$ tells us that, at least locally, $x > 0$ and $y > 0$. Thus $\ln(y) = 2\ln(x) + C$.

Moreover, plugging in $x = 1$ and $y = 2$, we find $C = \ln(2)$.

Solving $\ln(y) = 2\ln(x) + \ln(2)$ for y, we find $y = e^{2\ln(x)+\ln(2)} = 2x^2$.

Comment. When solving a DE or IVP, we can generally only expect to find a **local solution**, meaning that our solution might only be valid in a small interval around the initial condition (here, we can only expect $y(x)$ to be a solution for all x in an interval around 1; especially since we assumed $x > 0$ in our solution). However, we can check (do it!) that the solution $y = 2x^2$ is actually a **global solution** (meaning that it is a solution for all x, not just locally around 1).

Example 17. Solve the IVP $xy' = 2y$, $y(1) = -1$.

Solution. Again, we rewrite the DE as $\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{x}$, multiply both sides with $\mathrm{d}x$, and integrate to get $\int \frac{1}{y}\mathrm{d}y = \int \frac{2}{x}\mathrm{d}x$.

Hence, $\ln|y| = 2\ln|x| + C$. The initial condition $y(1) = -1$ tells us that, at least locally, $x > 0$ and $y < 0$ (note that this means $|y| = -y$). Thus $\ln(-y) = 2\ln(x) + C$.

Moreover, plugging in $x = 1$ and $y = -1$, we find $C = 0$.

Solving $\ln(-y) = 2\ln(x)$ for y, we find $y = -e^{2\ln(x)} = -x^2$. We easily verify that this is indeed a global solution.

Example 18. $y' = x + y$ is a DE for which the variables cannot be separated.

No worries, very soon we will have several tools to solve this DE as well.

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Existence and uniqueness of solutions

The following is a very general result that allows us to guarantee that "nice" IVPs must have a solution and that this solution is unique.

Comment. Note that any first-order DE can be written as $g(y', y, x) = 0$ where g is some function of three variables. Assuming that g is reasonable, we can solve for y' and rewrite such a DE as $y' = f(x, y)$ (for some, possibly complicated, function f).

Comment. To be precise, a solution to the IVP $y' = f(x, y)$, $y(a) = b$ is a function $y(x)$, defined on an interval I containing a, such that $y'(x) = f(x, y(x))$ for all $x \in I$ and $y(a) = b$.

Theorem 19. (existence and uniqueness) Consider the IVP $y' = f(x, y)$, $y(a) = b$.
If both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous [in a rectangle] around (a, b), then the IVP has a unique solution in some interval $x \in (a - \delta, a + \delta)$ where $\delta > 0$.

Comment. The interval around a might be very small. In other words, the δ in the theorem could be very small.

Comment. Note that the theorem makes two important assertions. First, it says that there exists a **local solution**. Second, it says that this solution is unique. These two parts of the theorem are famous results usually attributed to Peano (existence) and Picard–Lindelöf (uniqueness).

Advanced comment. The condition about $\frac{\partial}{\partial y} f(x, y)$ is a bit technical (and not optimal). If we drop this condition, we still get existence but, in general, no longer uniqueness.

Advanced comment. The interval in which the solution is unique could be smaller than the interval in which it exists. In other words, it is possible that, away from the initial condition, the solution "forks" into two or more solutions. Note that this does not contradict the theorem because it only guarantees uniqueness on a small interval.

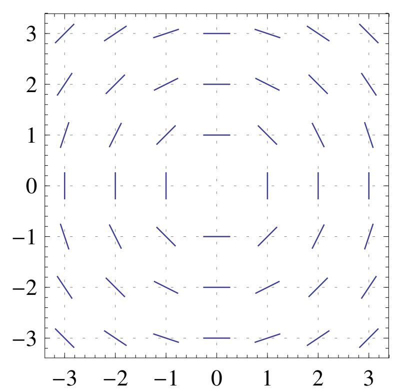
Example 20. Consider, again, the IVP $y' = -x/y$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = -x/y$.

We compute that $\frac{\partial}{\partial y} f(x, y) = x/y^2$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $y \neq 0$.

Hence, if $b \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.



Comment. In Example 13, we found that the DE $y' = -x/y$ is solved by $y(x) = \pm\sqrt{D - x^2}$.

Assume $b > 0$ (things work similarly for $b < 0$). Then $y(x) = \sqrt{D - x^2}$ solves the IVP (we need to choose D so that $y(a) = b$) if we choose $D = a^2 + b^2$. This confirms that there exists a solution. On the other hand, uniqueness means that there can be no other solution to the IVP than this one.

What happens in the case $b = 0$?

Solution. In this case, the existence and uniqueness theorem does not guarantee anything. If $a \neq 0$, then $y(x) = \sqrt{a^2 - x^2}$ and $y(x) = -\sqrt{a^2 - x^2}$ both solve the IVP (so we certainly don't have uniqueness), however only in a weak sense: namely, both of these solutions are not valid locally around $x = a$ but only in an interval of which a is an endpoint (for instance, the IVP $y' = -x/y$, $y(2) = 0$ is solved by $y(x) = \pm\sqrt{4 - x^2}$ but both of these solutions are only valid on the interval $[-2, 2]$ which ends at 2, and neither of these solutions can be extended past 2).

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Review. Existence and uniqueness theorem (Theorem 19) for an IVP $y' = f(x,y)$, $y(a) = b$:
If $f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$ are continuous around (a,b) then, locally, the IVP has a unique solution.

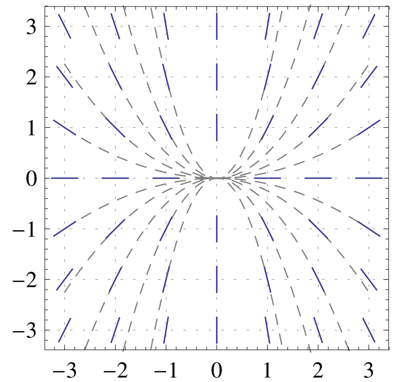
Example 21. Consider, again, the IVP $xy' = 2y$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x,y)$ with $f(x,y) = 2y/x$.

We compute that $\frac{\partial}{\partial y} f(x,y) = 2/x$.

We observe that both $f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$ are continuous for all (x,y) with $x \neq 0$.

Hence, if $a \neq 0$, then the IVP locally has a unique solution by the existence and uniqueness theorem.



What happens in the case $a = 0$?

Solution. In Example 15, we found that the DE $xy' = 2y$ is solved by $y(x) = Cx^2$.

This means that the IVP with $y(0) = 0$ has infinitely many solutions.

On the other hand, the IVP with $y(0) = b$ where $b \neq 0$ has no solutions. (This follows from the fact that there are no solutions to the DE besides $y(x) = Cx^2$. Can you see this by looking at the slope field?)

Example 22. Consider the IVP $y' = ky^2$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x,y)$ with $f(x,y) = ky^2$. We compute that $\frac{\partial}{\partial y} f(x,y) = 2ky$.

We observe that both $f(x,y)$ and $\frac{\partial}{\partial y} f(x,y)$ are continuous for all (x,y).

Hence, for any initial conditions, the IVP locally has a unique solution by the existence and uniqueness theorem.

Example 23. Solve $y' = ky^2$.

Solution. Separate variables to get $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = k$.

Integrating $\int \frac{1}{y^2} \mathrm{d}y = \int k \mathrm{d}x$, we find $-\frac{1}{y} = kx + C$.

We solve for y to get $y = -\frac{1}{C + kx} = \frac{1}{D - kx}$ (with $D = -C$). That is the solution we verified earlier!

Comment. Note that we did not find the solution $y = 0$ (it was "lost" when we divided by y^2). It is called a **singular solution** because it is not part of the **general solution** (the one-parameter family found above). However, note that we can obtain it from the general solution by letting $D \to \infty$.

Caution. We have to be careful about transforming our DE when using separation of variables: Just as the division by y^2 made us lose a solution, other transformations can add extra solutions which do not solve the original DE.

Here is a silly example (silly, because the transformation serves no purpose here) which still illustrates the point. The DE $(y-1)y' = (y-1)ky^2$ has the same solutions as $y' = ky^2$ plus the additional solution $y = 1$ (which does not solve $y' = ky^2$).

Example 24. (extra) Solve the IVP $y' = y^2$, $y(0) = 1$.

Solution. From the previous example with $k = 1$, we know that $y(x) = \frac{1}{D - x}$.

Using $y(0) = 1$, we find that $D = 1$ so that the unique solution to the IVP is $y(x) = \frac{1}{1 - x}$.

Comment. Note that we already concluded the uniqueness from the existence and uniqueness theorem.

On the other hand, note that $y(x) = \frac{1}{1-x}$ is only valid on $(-\infty, 1)$ and that it cannot be continuously extended past $x = 1$; it is only a local solution.

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Example 25. (homework) Consider the IVP $(x - y^2)y' = 3x$, $y(4) = b$. For which choices of b does the existence and uniqueness theorem guarantee a unique (local) solution?

Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = 3x/(x - y^2)$. We compute that $\frac{\partial}{\partial y} f(x, y) = 6xy/(x - y^2)^2$.
We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y) with $x - y^2 \neq 0$.
Note that $4 - b^2 \neq 0$ is equivalent to $b \neq \pm 2$.
Hence, if $b \neq \pm 2$, then the IVP locally has a unique solution by the existence and uniqueness theorem.

Linear first-order DEs

A **linear differential equation** is one where the function y and its derivatives only show up linearly (i.e. there are no terms such as y^2, $1/y$ or $\sin(y)$).
As such, the most general linear first-order DE is of the form

$$A(x)y' + B(x)y + C(x) = 0.$$

Comment. Note that any such DE can also be rewritten in the form $y' + P(x)y = Q(x)$ by dividing by $A(x)$ and rearranging. We will use this form when solving linear first-order DEs.

Example 26. (extra) Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2$.

Solution. (separation of variables) $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, $-\frac{1}{y} = x^2 + C$.
Hence the general solution is $y = \frac{1}{D - x^2}$. [There also is the singular solution $y = 0$.]

Solution. (in other words) Note that $\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ can be written as $\frac{\mathrm{d}}{\mathrm{d}x}\left[-\frac{1}{y}\right] = \frac{\mathrm{d}}{\mathrm{d}x}[x^2]$.
From there it follows that $-\frac{1}{y} = x^2 + C$, as above.

We now use the idea of writing both sides as a derivative to also solve linear DEs that are not separable.

The multiplication by $\frac{1}{y^2}$ will be replaced by multiplication with a so-called **integrating factor**.

Example 27. Solve $y' = x - y$.

Comment. Note that we cannot use separation of variables this time.

Solution. Rewrite the DE as $y' + y = x$.
Next, multiply both sides with e^x (we will see in a little bit how to find this "integrating factor") to get

$$\underbrace{e^x y' + e^x y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[e^x y]} = x e^x.$$

The "magic" part is that we are able to realize the new left-hand side as a derivative!
Next, we will integrate both sides and then solve for y. (Try it yourself!) To be continued...

Example 28. (resumed) Solve $y' = x - y$.

Comment. Note that we cannot use separation of variables this time.

Solution. Rewrite the DE as $y' + y = x$.
Next, multiply both sides with e^x (we will see in a little bit how to find this "integrating factor") to get

$$\underbrace{e^x y' + e^x y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[e^x y]} = x e^x.$$

The "magic" part is that we are able to realize the new left-hand side as a derivative!
We can then integrate both sides to get

$$e^x y = \int x e^x \mathrm{d}x = x e^x - e^x + C.$$

From here it follows that $y = x - 1 + Ce^{-x}$.

Comment. For the final integral, we used that $\int x e^x \mathrm{d}x = x e^x - \int e^x \mathrm{d}x = x e^x - e^x + C$ which follows, for instance, via integration by parts (with $f(x) = x$ and $g'(x) = e^x$ in the formula reviewed below).

Review. The multiplication rule $(fg)' = f'g + fg'$ implies $fg = \int f'g + \int fg'$.

The latter is equivalent to **integration by parts**:

$$\int f(x) g'(x) \mathrm{d}x = f(x) g(x) - \int f'(x) g(x) \mathrm{d}x$$

Comment. Sometimes, one writes $g'(x)\mathrm{d}x = \mathrm{d}g(x)$.

In general, we can solve any **linear first-order DE** $y' + P(x) y = Q(x)$ in this way.

- We want to multiply with an **integrating factor** $f(x)$ such that the left-hand side of the DE becomes

$$f(x) y' + f(x) P(x) y = \frac{\mathrm{d}}{\mathrm{d}x}[f(x) y].$$

  Since $\frac{\mathrm{d}}{\mathrm{d}x}[f(x) y] = f(x) y' + f'(x) y$, we need $f'(x) = f(x) P(x)$ for that.

- Check that $f(x) = \exp\left(\int P(x) \mathrm{d}x\right)$ has this property.

  Comment. This follows directly from computing the derivative of this $f(x)$ via the chain rule.

  Homework. On the other hand, note that finding f meant solving the DE $f' = P(x) f$. This is a separable DE. Solve it by separation of variables to arrive at the above formula for $f(x)$ yourself.

  Just to make sure. There is no difference between $\exp(x)$ and e^x. Here, we prefer the former notation for typographical reasons.

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With that integrating factor, we have the following recipe for solving any linear first-order equation:

(solving linear first-order DEs)

(a) Write the DE in the form $y' + P(x)y = Q(x)$.

(b) Compute the integrating factor as $f(x) = \exp\left(\int P(x)\mathrm{d}x\right)$.

[We can choose any constant of integration.]

(c) Multiply the DE from part (a) by $f(x)$ to get

$$\underbrace{f(x)y' + f(x)P(x)y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[f(x)y]} = f(x)Q(x).$$

(d) Integrate both sides to get

$$f(x)y = \int f(x)Q(x)\mathrm{d}x + C.$$

Then solve for y by dividing by $f(x)$.

Comment. For better understanding, we prefer to go through the above steps. On the other hand, we can combine these steps into the following formula for the general solution of $y' + P(x)y = Q(x)$:

$$y = \frac{1}{f(x)}\left(\int f(x)Q(x)\mathrm{d}x + C\right) \quad \text{where } f(x) = e^{\int P(x)\mathrm{d}x}$$

Existence and uniqueness. Note that the solution we construct exists on any interval on which P and Q are continuous (not just on some possibly very small interval). This is better than what the existence and uniqueness theorem (Theorem 19) can guarantee. This is one of the many ways in which linear DEs have particularly nice properties compared to DEs in general.

Example 29. Solve $x^2 y' = 1 - xy + 2x$, $y(1) = 3$.

Solution. This is a linear first-order DE. We can therefore solve it according to the recipe above.

(a) Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ with $P(x) = \frac{1}{x}$ and $Q(x) = \frac{1}{x^2} + \frac{2}{x}$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^{\ln x} = x$.

Here, we could write $\ln x$ instead of $\ln|x|$ because the initial condition tells us that $x > 0$, at least locally.

Comment. We can also choose a different constant of integration but that would only complicate things.

(c) Multiply the (rewritten) DE by $f(x) = x$ to get

$$\underbrace{x\frac{\mathrm{d}y}{\mathrm{d}x} + y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[xy]} = \frac{1}{x} + 2.$$

(d) Integrate both sides to get (again, we use that $x > 0$ to avoid having to use $|x|$)

$$xy = \int \left(\frac{1}{x} + 2\right)\mathrm{d}x = \ln x + 2x + C.$$

Using $y(1) = 3$ to find C, we get $1 \cdot 3 = \ln(1) + 2 \cdot 1 + C$ which results in $C = 3 - 2 = 1$.

Hence, the (unique) solution to the IVP is $y = \frac{\ln(x) + 2x + 1}{x}$.

Example 30. (extra) Solve $y' = 2y + 3x - 1$, $y(0) = 2$.

Solution. This is a linear first-order DE.

(a) Rewrite the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$ with $P(x) = -2$ and $Q(x) = 3x - 1$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)\mathrm{d}x\right) = e^{-2x}$.

(c) Multiply the (rewritten) DE by $f(x) = e^{-2x}$ to get

$$\underbrace{e^{-2x}\frac{\mathrm{d}y}{\mathrm{d}x} - 2e^{-2x}y}_{= \frac{\mathrm{d}}{\mathrm{d}x}[e^{-2x}y]} = (3x-1)e^{-2x}.$$

(d) Integrate both sides to get

$$\begin{aligned} e^{-2x}y &= \int (3x-1)e^{-2x}\mathrm{d}x \\ &= 3\int x e^{-2x}\mathrm{d}x - \int e^{-2x}\mathrm{d}x \\ &= 3\left(-\frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x}\right) - \left(-\frac{1}{2}e^{-2x}\right) + C \\ &= -\frac{3}{2}x e^{-2x} - \frac{1}{4}e^{-2x} + C. \end{aligned}$$

Here, we used that $\int x e^{-2x}\mathrm{d}x = -\frac{1}{2}x e^{-2x} + \frac{1}{2}\int e^{-2x}\mathrm{d}x = -\frac{1}{2}x e^{-2x} - \frac{1}{4}e^{-2x}$ (for instance, via integration by parts with $f(x) = x$ and $g'(x) = e^{-2x}$).

Hence, the general solution is $y(x) = -\frac{3}{2}x - \frac{1}{4} + Ce^{2x}$.

Solving $y(0) = -\frac{1}{4} + C = 2$ for C yields $C = \frac{9}{4}$.

In conclusion, the (unique) solution to the IVP is $y(x) = -\frac{3}{2}x - \frac{1}{4} + \frac{9}{4}e^{2x}$.

Substitutions in DEs

Example 31. (review) Using substitution, compute $\int \frac{x}{1+x^2}\mathrm{d}x$.

Solution. We substitute $t = 1 + x^2$. In that case, $\mathrm{d}t = 2x\,\mathrm{d}x$.
$\int \frac{x}{1+x^2}\mathrm{d}x = \frac{1}{2}\int \frac{1}{t}\mathrm{d}t = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln(1+x^2) + C$

Comment. Why were we allowed to drop the absolute value in the logarithm?

Review. On the other hand, recall that $\int \frac{1}{1+x^2}\mathrm{d}x = \arctan(x) + C$.

Example 32. Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = (x+y)^2$.

First things first. Is this DE separable? Is it linear? (No to both but make sure that this is clear to you.) This means that our previous techniques are not sufficient to solve this DE.

Solution. Looking at the right-hand side, we have a feeling that the substitution $u = x + y$ might simplify things. Then $y = u - x$ and, therefore, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} - 1$.

Using these, the DE translates into $\frac{\mathrm{d}u}{\mathrm{d}x} - 1 = u^2$. This is a separable DE: $\frac{1}{1+u^2}\mathrm{d}u = \mathrm{d}x$

After integration, we find $\arctan(u) = x + C$ and, thus, $u = \tan(x + C)$.

The solution of the original DE is $y = u - x = \tan(x + C) - x$.

Example 33. Consider the DE $x\frac{\mathrm{d}y}{\mathrm{d}x} = y + y^2 f(x)$.

(a) Substitute $u = \frac{y}{x}$. Is the resulting DE separable or linear?

(b) Substitute $v = \frac{1}{y}$. Is the resulting DE separable or linear?

(c) **(homework)** Solve each of the new DEs.

Solution.

(a) Set $u = \frac{y}{x}$. Then $y = ux$ and, thus, $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u$.

Using these, the DE translates into $x\left(x\frac{\mathrm{d}u}{\mathrm{d}x} + u\right) = ux + (ux)^2 f(x)$.

This DE simplifies to $\frac{\mathrm{d}u}{\mathrm{d}x} = u^2 f(x)$. This is a separable DE.

(b) Set $v = \frac{1}{y}$. Then $y = \frac{1}{v}$ and, thus, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x}$.

Using these, the DE translates into $x\left(-\frac{1}{v^2}\frac{\mathrm{d}v}{\mathrm{d}x}\right) = \frac{1}{v} + \frac{1}{v^2} f(x)$.

This DE simplifies to $x\frac{\mathrm{d}v}{\mathrm{d}x} = -v - f(x)$. This is a linear DE.

(c) Let us write $F(x)$ for an antiderivative of $f(x)$.

- The DE $\frac{\mathrm{d}u}{\mathrm{d}x} = u^2 f(x)$ from the first part is separable: $u^2 \mathrm{d}u = f(x)\mathrm{d}x$.
  After integration, we find $-\frac{1}{u} = F(x) + C$.
  Since $u = \frac{y}{x}$, this becomes $-\frac{x}{y} = F(x) + C$.
  The general solution of the initial DE therefore is $y = -\frac{x}{F(x) + C}$.
- The DE $x\frac{\mathrm{d}v}{\mathrm{d}x} = -v - f(x)$ from the second part is linear. We apply our recipe:

  (a) Rewrite the DE as $\frac{\mathrm{d}v}{\mathrm{d}x} + P(x)v = Q(x)$ with $P(x) = 1/x$ and $Q(x) = -f(x)/x$.

  (b) The integrating factor is $\exp\left(\int P(x)\mathrm{d}x\right) = e^{\ln x} = x$.

  Comment. We should make a mental note that we assumed that $x > 0$. In the next step, however, we see that the integrating factor works for all x.

  (c) Multiply the (rewritten) DE by the integrating factor x to get $\underbrace{x\frac{\mathrm{d}v}{\mathrm{d}x} + v}_{= \frac{\mathrm{d}}{\mathrm{d}x}[xv]} = -f(x)$.

  (d) Integrate both sides to get $xv = -F(x) + C$.

  Since $v = \frac{1}{y}$, we find $\frac{x}{y} = -F(x) + C$.
  The general solution of the initial DE therefore is $y = -\frac{x}{F(x) - C}$.

Comment. Note that our two approaches led to the same general solution (from the existence and uniqueness theorem, we can see that this must be the case). One of the formulas features $+C$ while the other features $-C$. However, that makes no difference because C is a free parameter (we could have given them different names if we preferred).

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Useful substitutions

The previous example illustrates that different substitutions can help to solve a given DE.

Choosing the right substitution is difficult in general. The following is a compilation of important cases that are easy to spot and for which the listed substitutions are guaranteed to succeed:

- $y' = F\left(\frac{y}{x}\right)$. This is called a **homogeneous equation**.

  Set $u = \frac{y}{x}$. Then $y = ux$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}u}{\mathrm{d}x} + u$. We get $x\frac{\mathrm{d}u}{\mathrm{d}x} + u = F(u)$. This DE is always separable.

  Caution. We will soon discuss homogeneous linear differential equations, where the label homogeneous means something different (though in both cases, there is a common underlying reason).

- $y' = F(ax + by)$

  Set $u = ax + by$. Then $y = \frac{1}{b}(u - ax)$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{b}\left(\frac{\mathrm{d}u}{\mathrm{d}x} - a\right)$.

  The new DE is $\frac{1}{b}\left(\frac{\mathrm{d}u}{\mathrm{d}x} - a\right) = F(u)$ or, simplified, $\frac{\mathrm{d}u}{\mathrm{d}x} = a + bF(u)$. This DE is always separable.

- $y' = F(x)y + G(x)y^n$. This is called a **Bernoulli equation**.

  Set $u = y^{1-n}$. The resulting DE is always linear. We will consider this case next time.

- $F(y'', y', x) = 0$ (2nd order with "y missing")

  Set $u = y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. Then $y'' = \frac{\mathrm{d}u}{\mathrm{d}x}$. We get the first-order DE $F\left(\frac{\mathrm{d}u}{\mathrm{d}x}, u, x\right) = 0$.

- $F(y'', y', y) = 0$ (2nd order with "x missing")

  Set $u = y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. Then $y'' = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot u$. We get the first-order DE $F\left(u\frac{\mathrm{d}u}{\mathrm{d}y}, u, y\right) = 0$.

Example 34. (homework) Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = (2x - 3y)^2 + \frac{2}{3}$, $y(1) = \frac{1}{3}$.

Solution. This is of the form $y' = F(2x - 3y)$ with $F(t) = t^2 + \frac{2}{3}$.

Therefore, as suggested by the above list, we substitute $u = 2x - 3y$.

Then $y = \frac{1}{3}(2x - u)$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}\left(2 - \frac{\mathrm{d}u}{\mathrm{d}x}\right)$.

The new DE is $\frac{1}{3}\left(2 - \frac{\mathrm{d}u}{\mathrm{d}x}\right) = u^2 + \frac{2}{3}$ or, simplified, $\frac{\mathrm{d}u}{\mathrm{d}x} = -3u^2$.

This DE is separable: $u^{-2}\mathrm{d}u = -3\mathrm{d}x$. After integration, $-\frac{1}{u} = -3x + C$.

We conclude that $u = \frac{1}{3x - C}$ and, hence, $y(x) = \frac{1}{3}(2x - u) = \frac{2}{3}x - \frac{1}{3}\frac{1}{3x - C}$.

Solving $y(1) = \frac{2}{3} - \frac{1}{3}\frac{1}{3 - C} = \frac{1}{3}$ for C leads to $C = 2$.

Hence, the unique solution of the IVP is $y(x) = \frac{2}{3}x - \frac{1}{3(3x - 2)}$.

Example 35. Consider $\frac{dy}{dx} = F(x)y + G(x)y^n$. This is called a **Bernoulli equation**. Substitute $u = y^{1-n}$ and show that the resulting linear DE.

Solution. If $u = y^{1-n}$ then $y = u^{1/(1-n)}$ and, thus, $\frac{dy}{dx} = \frac{1}{1-n}u^{n/(1-n)}\frac{du}{dx}$. $\left[\frac{1}{1-n} - 1 = \frac{n}{1-n}\right]$

The new DE is $\frac{1}{1-n}u^{n/(1-n)}\frac{du}{dx} = F(x)u^{1/(1-n)} + G(x)u^{n/(1-n)}$.

Dividing both sides by $u^{n/(1-n)}$, the DE simplifies to $\frac{1}{1-n}\frac{du}{dx} = F(x)u + G(x)$.

Comment. The original DE has the trivial solution $y = 0$. Do you see where we might lose that solution?

Example 36. (homework) Solve the IVP $\frac{dy}{dx} = 2y - 3xy^5$, $y(0) = 1$.

Solution. This is an example of a Bernoulli equation (with $n = 5$). We therefore substitute $u = y^{1-n} = y^{-4}$. Accordingly, $y = u^{-1/4}$ and, thus, $\frac{dy}{dx} = -\frac{1}{4}u^{-5/4}\frac{du}{dx}$.

The new DE is $-\frac{1}{4}u^{-5/4}\frac{du}{dx} = 2u^{-1/4} - 3xu^{-5/4}$, which simplifies to $\frac{du}{dx} = -8u + 12x$.

This is a linear first-order DE, which we solve according to our recipe:

(a) Rewrite the DE as $\frac{du}{dx} + P(x)u = Q(x)$ with $P(x) = 8$ and $Q(x) = 12x$.

(b) The integrating factor is $f(x) = \exp\left(\int P(x)dx\right) = e^{8x}$.

(c) Multiply the (rewritten) DE by $f(x) = e^{8x}$ to get

$$\underbrace{e^{8x}\frac{du}{dx} + 8e^{8x}u}_{=\frac{d}{dx}[e^{8x}u]} = 12xe^{8x}.$$

(d) Integrate both sides to get:

$$e^{8x}u = 12\int xe^{8x}dx = 12\left(\frac{1}{8}xe^{8x} - \frac{1}{8^2}e^{8x}\right) + C = \frac{3}{2}xe^{8x} - \frac{3}{16}e^{8x} + C$$

Here we used that $\int xe^{ax}dx = \frac{1}{a}xe^{ax} - \frac{1}{a^2}e^{ax}$. (Integration by parts!)

The general solution of the DE for u therefore is $u = \frac{3}{2}x - \frac{3}{16} + Ce^{-8x}$.

Correspondingly, the general solution of the initial DE is $y = u^{-1/4} = 1/\sqrt[4]{\frac{3}{2}x - \frac{3}{16} + Ce^{-8x}}$.

Using $y(0) = 1$, we find $1 = 1/\sqrt[4]{C - \frac{3}{16}}$ from which we obtain $C = 1 + \frac{3}{16} = \frac{19}{16}$.

The unique solution to the IVP therefore is $y = 1/\sqrt[4]{\frac{3}{2}x - \frac{3}{16} + \frac{19}{16}e^{-8x}}$.

Example 37. Solve $(x-y)\frac{dy}{dx}=x+y$.

Solution. Divide the DE by x to get $(1-\frac{y}{x})\frac{dy}{dx}=1+\frac{y}{x}$. This is a homogeneous equation!

We therefore substitute $u=\frac{y}{x}$. Then $y=ux$ and $\frac{dy}{dx}=x\frac{du}{dx}+u$.

The resulting DE is $(x-ux)\left(x\frac{du}{dx}+u\right)=x+ux$, which simplifies to $x(1-u)\frac{du}{dx}=1+u^2$.

This DE is separable: $\frac{1-u}{1+u^2}du=\frac{1}{x}dx$

Integrating both sides, we find $\arctan(u)-\frac{1}{2}\ln(1+u^2)=\ln|x|+C$.

Setting $u=y/x$, we get the (general) implicit solution $\arctan(y/x)-\frac{1}{2}\ln(1+(y/x)^2)=\ln|x|+C$.

Comment. We used $\int\frac{1}{1+u^2}du=\arctan(u)+C$ and $\int\frac{x}{1+x^2}dx=\frac{1}{2}\ln(1+x^2)+C$ when integrating.

See Example 31 where we reviewed these integrals.

Solving simple 2nd order DEs

We have the following two useful substitutions for certain simple DEs of order 2:

- $F(y'',y',x)=0$ (2nd order with "y missing")

  Set $u=y'=\frac{dy}{dx}$. Then $y''=\frac{du}{dx}$. We get the first-order DE $F\left(\frac{du}{dx},u,x\right)=0$.

- $F(y'',y',y)=0$ (2nd order with "x missing")

  Set $u=y'=\frac{dy}{dx}$. Then $y''=\frac{du}{dx}=\frac{du}{dy}\cdot\frac{dy}{dx}=\frac{du}{dy}\cdot u$. We get the first-order DE $F\left(u\frac{du}{dy},u,y\right)=0$.

Example 38. Solve $y''=x-y'$.

Solution. We substitute $u=y'$, which results in the first-order DE $u'=x-u$.

This DE is linear and, using our recipe (see below for the details), we can solve it to find $u=x-1+Ce^{-x}$.

Since $y'=u$, we conclude that the general solution is $y=\int(x-1+Ce^{-x})dx=\frac{1}{2}x^2-x-Ce^{-x}+D$.

Important comment. This is a DE of order 2. Hence, as expected, the general solution has two free parameter.

Solving the linear DE. To solve $u'=x-u$ (also see Example 28, where we had solved this DE before), we

(a) rewrite the DE as $\frac{du}{dx}+P(x)u=Q(x)$ with $P(x)=1$ and $Q(x)=x$.

(b) The integrating factor is $f(x)=\exp\left(\int P(x)dx\right)=e^x$.

(c) Multiply the (rewritten) DE by $f(x)=e^x$ to get $\underbrace{e^x\frac{du}{dx}+e^x u}_{=\frac{d}{dx}[e^x u]}=xe^x$.

(d) Integrate both sides to get (using integration by parts): $e^x u=\int xe^x dx=xe^x-e^x+C$

Hence, the general solution of the DE for u is $u=x-1+Ce^{-x}$, which is what we used above.

Example 39. (homework) Solve the IVP $y''=x-y'$, $y(0)=1$, $y'(0)=2$.

Solution. As in the previous example, we find that the general solution to the DE is $y(x)=\frac{1}{2}x^2-x-Ce^{-x}+D$.

Using $y'(x)=x-1+Ce^{-x}$ and $y'(0)=2$, we find that $2=-1+C$. Hence, $C=3$.

Then, using $y(x)=\frac{1}{2}x^2-x-3e^{-x}+D$ and $y(0)=1$, we find $1=-3+D$. Hence, $D=4$.

In conclusion, the unique solution to the IVP is $y(x)=\frac{1}{2}x^2-x-3e^{-x}+4$.

Example 40. **(extra)** Find the general solution to $y'' = 2yy'$.

Solution. We substitute $u = y' = \frac{\mathrm{d}y}{\mathrm{d}x}$. Then $y'' = \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}y} \cdot u$.

Therefore, our DE turns into $u\frac{\mathrm{d}u}{\mathrm{d}y} = 2yu$.

Dividing by u, we get $\frac{\mathrm{d}u}{\mathrm{d}y} = 2y$. [Note that we lose the solution $u = 0$, which gives the singular solution $y = C$.]

Hence, $u = y^2 + C$. It remains to solve $y' = y^2 + C$. This is a separable DE.

$\frac{1}{C + y^2}\mathrm{d}y = \mathrm{d}x$. Let us restrict to $C = D^2 \geqslant 0$ here. (This means we will only find "half" of the solutions.)

$\int \frac{1}{D^2 + y^2}\mathrm{d}y = \frac{1}{D^2}\int \frac{1}{1 + (y/D)^2}\mathrm{d}y = \frac{1}{D}\arctan(y/D) = x + A$.

Solving for y, we find $y = D\tan(Dx + AD) = D\tan(Dx + B)$. [$B = AD$]

Applications of DEs & Modeling

The exponential model of population growth

If $P(t)$ is the size of a population (eg. of bacteria) at time t, then the rate of change $\frac{\mathrm{d}P}{\mathrm{d}t}$ might, from biological considerations, be (nearly) proportional to $P(t)$.

Comment. "Population" might sound more specific than it is. It could also refer to rather different populations such as amounts of money (finance) or amounts of radioactive material (physics).

For instance, thinking about an amount $P(t)$ of money in a bank account at time t, we would also expect $\frac{\mathrm{d}P}{\mathrm{d}t}$ (the money per time that we gain from receiving interest) to be proportional to $P(t)$.

The corresponding **mathematical model** is described by the DE $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ where k is the constant of proportionality.

Example 41. Determine all solutions to the DE $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$.

Solution. We easily guess and then verify that $P(t) = Ce^{kt}$ is a solution. (Alternatively, we can find this solution via separation of variables or because this is a linear DE. Do it both ways!)

Moreover, it follows from the existence and uniqueness theorem that there cannot be further solutions. (Alternatively, we can conclude this from our solving process (separation of variables or our approach to linear DEs only lose solutions when we divide by zero and we can consider those cases separately)).

Mathematics therefore tells us that the (only) solutions to this DE are given by $P(t) = Ce^{kt}$ where C is some constant.

Hence, populations satisfying the assumption from biology necessarily exhibit exponential growth.

Example 42. Let $P(t)$ describe the size of a population at time t. Suppose $P(0) = 100$ and $P(1) = 300$. Under the exponential model of population growth, find $P(t)$.

Solution. $P(t)$ solves the DE $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ and therefore is of the form $P(t) = Ce^{kt}$.

We now use the two data points to determine both C and k.

$Ce^{k \cdot 0} = C = 100$ and $Ce^k = 100e^k = 300$. Hence $k = \ln(3)$ and $P(t) = 100e^{\ln(3)t} = 100 \cdot 3^t$.

Main problem of modeling: a model has to be detailed enough to resemble the real world, yet simple enough to allow for mathematical analysis.

The logistic model of population growth

If the population is constrained by resources, then $\frac{\mathrm{d}P}{\mathrm{d}t} = kP$ is not a good model. A model to take that into account is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$. This is the **logistic equation**.

M is called the carrying capacity:

- Note that if $P \ll M$ then $1 - \frac{P}{M} \approx 1$ and we are back to the simpler exponential model. This means that the population P will grow (nearly) exponentially if P is much less than the carrying capacity M.

- On the other hand, if $P > M$ then $1 - \frac{P}{M} < 0$ so that (assuming $k > 0$) $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$, which means that the population P is shrinking if it exceeds the carrying capacity M.

Comment. If $P(t)$ is the size of a population, then P'/P can be interpreted as its *per capita growth rate*.
Note that in the exponential model we have that $P'/P = k$ is constant.
On the other hand, in the logistic model we have that $P'/P = k(1 - P/M)$ is a linear function.

Example 43. Solve the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$.

Solution. This is a separable DE: $\frac{1}{P(1 - \frac{P}{M})}\mathrm{d}P = k\mathrm{d}t$.

To integrate the left-hand side, we use partial fractions: $\frac{1}{P(1 - \frac{P}{M})} = \frac{1}{P} + \frac{1/M}{1 - \frac{P}{M}} = \frac{1}{P} - \frac{1}{P - M}$.

After integrating, we obtain $\ln|P| - \ln|P - M| = kt + A$.

Equivalently, $\ln\left|\frac{P}{P - M}\right| = kt + A$ so that $\frac{P}{P - M} = \pm e^{kt + A} = Be^{kt}$ where $B = \pm e^A$.

Solving for P, we conclude that the general solution is

$$P(t) = \frac{BMe^{kt}}{Be^{kt} - 1} = \frac{M}{1 + Ce^{-kt}},$$

where replaced the free parameter B with $C = -1/B$.

Initial population. Note that the initial population is $P(0) = \frac{M}{1 + C}$. Equivalently, $C = \frac{M}{P(0)} - 1$ which expresses the free parameter C in terms of the initial population.

Comment. Note that $B = \pm e^A$ can be any real number except 0. However, we can easily check that $B = 0$ also gives us a solution to the DE (namely, the trivial solution $P = 0$). This solution was "lost" when we divided by P to separate variables.

Exercise. Note that the logistic equation is a Bernoulli equation. As an alternative to separation of variables, we can therefore solve it by transforming it to a linear DE.

Review of partial fractions. Recall that partial fractions tells us that fractions like $\frac{p(x)}{(x - r_1)(x - r_2)\cdots}$ (with the numerator of smaller degree than the denominator; and with the r_j distinct) can be written as a sum of terms of the form $\frac{A_j}{x - r_j}$ for suitable constants A_j.

In our case, this tells us that $\frac{1}{P(1 - P/M)} = \frac{A}{P} + \frac{B}{1 - P/M}$ for certain constants A and B.

Multiply both sides by P and set $P = 0$ to find $A = 1$.

Multiply both sides by $1 - P/M$ and set $P = M$ to find $B = 1/M$. This is what we used above.

The **logistic equation** with growth rate k and carrying capacity M is

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right).$$

The general solution is $P(t) = \frac{M}{1 + Ce^{-kt}}$ where $C = \frac{M}{P(0)} - 1$.

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Example 44. Let $P(t)$ describe the size of a population at time t. Under the logistic model of population growth, what is $\lim_{t\to\infty} P(t)$?

Solution.

- If $k>0$, then $e^{-kt}\to 0$ and it follows from $P(t)=\frac{M}{1+Ce^{-kt}}$ that $\lim_{t\to\infty} P(t)=M$.
  In other words, the population will approach the carrying capacity in the long run.
- If $k=0$, then we simply have $P(t)=\frac{M}{1+C}$. In other words, the population remains constant.
  This is a corner case because the DE becomes $\frac{\mathrm{d}P}{\mathrm{d}t}=0$.
- If $k<0$, then $e^{-kt}\to\infty$ and it follows that $\lim_{t\to\infty} P(t)=0$.
  In other words, the population will approach extinction in the long run.

Example 45. (homework) A rising population is modeled by the equation $\frac{\mathrm{d}P}{\mathrm{d}t}=400P-2P^2$.

(a) When the population size stabilizes in the long term, how big will the population be?

(b) Under which condition will the population size shrink?

(c) What is the population size when it is growing the fastest?

(d) If $P(0)=10$, what is $P(t)$?

Solution.

(a) Once the population reaches a stable level in the long term, we have $\frac{\mathrm{d}P}{\mathrm{d}t}=0$ (no change in population size).
Hence, $0=400P-2P^2=2P(200-P)$ which implies that $P=0$ or $P=200$. Since the population is rising, it will approach 200 in the long term.
Alternatively. Our DE matches the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t}=kP\left(1-\frac{P}{M}\right)$ with $k=400$ and $M=200$.

(b) The population size will shrink if $\frac{\mathrm{d}P}{\mathrm{d}t}<0$.
The DE tells us that is the case if and only if $400P-2P^2<0$ or, equivalently, if $P>\frac{400}{2}=200$.
Comment. In the logistic model, the population shrinks if it exceeds the carrying capacity.

(c) This is asking when $\frac{\mathrm{d}P}{\mathrm{d}t}$ (the population growth) is maximal.
The DE is telling us that this growth is $f(P)=400P-2P^2$. This a parabola that opens to the bottom. From Calculus, we know that it has a global maximum when $f'(P)=0$.
$f'(P)=400-4P=0$ leads to $P=100$.
Thus, the population is growing the fastest when its size is 100.
Comment. In the logistic model, the population is growing fastest when it is half the carrying capacity.

(d) We know that the general solution of the logistic equation is $P(t)=\frac{M}{1+Ce^{-kt}}=\frac{200}{1+Ce^{-400t}}$.
Using $P(0)=10$, we find that $C=\frac{200}{10}-1=19$.
Thus $P(t)=\frac{200}{1+19e^{-400t}}$.

Example 46. A scientist is claiming that a certain population $P(t)$ follows the logistic model of population growth perfectly. How many data points do you need to begin to verify that claim?

Solution. The general solution $P(t)=\frac{M}{1+Ce^{-kt}}$ to the logistic equation has 3 parameters.

Hence, we need 3 data points just to solve for their values.

Once we have 4 or more data points, we are able to test whether $P(t)$ conforms to the logistic model.

Important comment. Complicated models tend to have many degrees of freedom, which makes it easier to fit them to real world data (even if the model is not actually particularly appropriate). We therefore need to be cognizant about how much evidence is needed to decide that a given model is appropriate for the data.

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Further population models

Let $P(t)$ be the size of the population that we wish to model at time t.

Denote with $\beta(t)$ and $\delta(t)$ the birth and death rate at time t, measured in number of births or deaths per unit of population per unit of time.

In the time interval $[t, t + \Delta t]$, we have that

$$\Delta P \approx \beta(t) P(t) \Delta t - \delta(t) P(t) \Delta t.$$

Comment. The reason that this is not an exact equation is that the rates $\beta(t)$ and $\delta(t)$ are allowed to change with t. In the above, we used these rates at time t for all times in $[t, t + \Delta t]$. This is a good approximation if Δt is small.

Divide both sides by Δt and let $\Delta t \to 0$ to obtain the general differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta(t) - \delta(t)) P.$$

Given certain scenarios, we now make corresponding reasonable choices for $\beta(t)$ and $\delta(t)$.

- **(basic)** If the rates $\beta(t)$ and $\delta(t)$ are constant over time, the DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta - \delta) P$.
  This is the exponential model of population growth.

- **(limited supply)** If supply is limited, the birth rate will decrease as P increases. The simplest such relationship would be a linear dependence, which would take the form $\beta(t) = \beta_0 - \beta_1 P$.
  On the other hand, we still assume that $\delta(t)$ is constant. (However, depending on circumstances, it could also be reasonable to assume that $\delta(t)$ increases as P increases.)
  With these assumptions, the corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta_0 - \beta_1 P - \delta) P$.
  This is the logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(1 - P/M)$ with $k = \beta_0 - \delta$ and $\frac{k}{M} = \beta_1$.

- **(rare isolated species)** If the population consists of rare and isolated specimen which rely on chance encounters to reproduce, then it is reasonable to assume that the birth rate $\beta(t)$ is proportional to $P(t)$ (larger $P(t)$ means more possibilities for chance encounters). Once more, we assume that $\delta(t)$ constant.
  With these assumptions, the corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = (kP - \delta) P$.
  This is, again, the logistic equation.

- **(rare isolated species with very long life)** As before, for a rare isolated population, it is reasonable to assume that $\beta(t)$ is proportional to $P(t)$. If, in addition, our specimen have very long life, then we would assume that $\delta(t) = 0$.
  The corresponding DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP^2$. Solutions are $P(t) = \frac{1}{C - kt}$ where $P(0) = 1/C$. (Do it!)
  Comment. Note that $P(t) \to \infty$ as $t \to C/k$. This explosion (which implies population growth beyond exponential growth) emphasizes that we can only use the DE while our initial assumptions are satisfied. Here, the DE is no longer appropriate when our species is no longer rare because $P(t)$ is too large.

- **(spread of contagious incurable virus)** Let $P(t)$ count the number of infected population units among a (constant) total of N. Since the virus is incurable, we have $\delta(t) = 0$. On the other hand, it is reasonable to assume that $\beta(t)$ is proportional to $N - P$ (the number of people that can still be infected).
  The resulting DE is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP(N - P)$. Once again, this is the logistic equation.

- **(harvesting)** Suppose that h population units are harvested each unit of time.
  Then the DE becomes $\frac{\mathrm{d}P}{\mathrm{d}t} = (\beta(t) - \delta(t)) P - h$.
  For instance. $\frac{\mathrm{d}P}{\mathrm{d}t} = kP - h$ has the solution $P(t) = Ce^{kt} + h/k$. In that case, we get exponential growth if $C > 0$. Note that $P(0) = C + h/k$. In terms of the initial population $P(0)$, we therefore get exponential growth if $P(0) > h/k$.

Review. The **logistic equation** is $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{M}\right)$.

Here, k is the growth rate and M is the carrying capacity.

The general solution is $P(t) = \frac{M}{1 + Ce^{-kt}}$ where $C = \frac{M}{P(0)} - 1$.

Example 47. In a city with a fixed population N, the time rate of change of the number P of people who have heard a certain rumor is proportional to the product of P and $N - P$. Suppose initially 10% have heard the rumor and after a week this number has grown to 20%. What percentage will this number reach after one more week?

Solution. $\frac{\mathrm{d}P}{\mathrm{d}t} = \gamma P(N - P)$. $P(0) = 0.1N$ and $P(1) = 0.2N$. We need $P(2)$.

Note that this is a logistic equation $\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{N}\right)$ with $k = \gamma N$ and carrying capacity N.

It therefore has the general solution $P(t) = \frac{N}{1 + Ce^{-kt}}$.

Using $P(0) = \frac{N}{1 + C} = 0.1N$, we find that $C = 9$.

Using $P(1) = \frac{N}{1 + 9e^{-k}} = 0.2N$, we further find that $e^{-k} = \frac{4}{9}$.

We could solve for k but note that it is more pleasing to use $e^{-kt} = (e^{-k})^t = \left(\frac{4}{9}\right)^t$ in our formula for $P(t)$.

We conclude that $P(t) = \frac{N}{1 + 9\left(\frac{4}{9}\right)^t}$.

In particular, $P(2) = \frac{N}{1 + 9 \cdot \frac{16}{81}} = \frac{9}{25}N$ which is 36%.

Mixing problems

Example 48. A tank contains 20gal of pure water. It is filled with brine (containing 5lb/gal salt) at a rate of 3gal/min. At the same time, well-mixed solution flows out at a rate of 2gal/min. How much salt is in the tank after t minutes?

Solution. Let $x(t)$ denote the amount of salt (in lb) in the tank after time t (in min).

At time t, the concentration of salt (in lb/gal) in the tank is $\frac{x(t)}{V(t)}$ where $V(t) = 20 + (3 - 2)t = 20 + t$ is the volume (in gal) in the tank.

In the time interval $[t, t + \Delta t]$: $\Delta x \approx 3 \cdot 5 \cdot \Delta t - 2 \cdot \frac{x(t)}{V(t)} \cdot \Delta t$.

Hence, $x(t)$ solves the IVP $\frac{\mathrm{d}x}{\mathrm{d}t} = 15 - 2 \cdot \frac{x}{20 + t}$ with $x(0) = 0$. Since this is a linear DE, we can solve it as follows:

- We write it in the form $\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{2}{20 + t}x = 15$.
- The integrating factor is $f(t) = \exp\left(\int \frac{2}{20 + t}\mathrm{d}t\right) = (20 + t)^2$.
- Multiply the (rewritten) DE by $f(t) = (20 + t)^2$ to get $\underbrace{(20 + t)^2\frac{\mathrm{d}x}{\mathrm{d}t} + 2(20 + t)x}_{= \frac{\mathrm{d}}{\mathrm{d}t}[(20+t)^2 x]} = 15(20 + t)^2$.
- Integrate both sides to get $(20 + t)^2 x = 15\int (20 + t)^2 \mathrm{d}t = 5(20 + t)^3 + C$.

Hence the general solution to the DE is $x(t) = 5(20 + t) + \frac{C}{(20 + t)^2}$. Using $x(0) = 0$, we find $C = -5 \cdot 20^3$.

We conclude that, after t minutes, the tank contains $x(t) = 5(20 + t) - \frac{5 \cdot 20^3}{(20 + t)^2}$ pounds of salt.

Comment. As a consequence, $x(t) \approx 5(20 + t) = 5V(t)$ for large t. Why does that make perfect sense?!

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Acceleration–velocity models

To model a falling object, we let $y(t)$ be its height at time t.

Then physics has names for $y'(t)$ and $y''(t)$: these are the **velocity** and the **acceleration**.

Physics tells us that objects fall due to gravity (and that it makes already falling objects fall faster; in other words, gravity accelerates falling objects). Physicists have measured that, on earth, the the graviational acceleration is $g \approx 9.81 \mathrm{m/s}^2$.

If we only take earth's gravitation into account, then the fall is therefore modelled by

$$y''(t) = -g.$$

Example 49. A ball is dropped from a 100m tall building. How long until it reaches the ground? What is the speed when it hits the ground?

Solution. Let $y(t)$ be the height (in meters) at which the ball is at time t (in seconds).

As above, physics tells us that an object falling due to gravity (and ignoring everything else) satisfies the DE $y'' = -g$ where $g \approx 9.81$. We further know the initial values $y(0) = 100$, $y'(0) = 0$.

Substituting $v = y'$ in the DE, we get $v' = -g$. This DE is solved by $v(t) = -gt + C$.

Hence, $y(t) = \int v(t)\mathrm{d}t = -\frac{1}{2}gt^2 + Ct + D$.

The initial conditions $y(0) = 100$, $y'(0) = 0$ tell us that $D = 100$ and $C = 0$.

Thus $y(t) = -\frac{1}{2}gt^2 + 100$.

The ball reaches the ground when $y(t) = -\frac{1}{2}gt^2 + 100 = 0$, that is after $t = \sqrt{200/g} \approx 4.52$ seconds.

The speed then is $|y'(4.5)| \approx 44.3\,m/s$.

For many applications, one needs to take air resistance into account.

This is actually less well understood than one might think, and the physics quickly becomes rather complicated. Typically, air resistance is somewhere in between the following two cases:

- Under certain assumptions, physics suggests that air resistance is proportional to the square of the velocity.

  Comment. A simplistic way to think about this is to imagine the falling object to bump into (air) particles; if the object falls twice as fast, then the momentum of the particles it bumps into is twice as large and it bumps into twice as many of them.

- In other cases such as "relatively slowly" falling objects, one might empirically observe that air resistance is proportional to the velocity itself.

  Comment. One might imagine that, at slow speed, the falling object doesn't exactly bump into particles but instead just gently pushes them aside; so that at twice the speed it only needs to gently push twice as often.

Example 50. When modeling the (slow) fall of a parachute, physics suggests that the air resistance is roughly proportional to velocity. If $y(t)$ is the parachute's height at time t, then the corresponding DE is $y'' = -g - \rho y'$ where $\rho > 0$ is a constant.

Comment. Note that $-\rho y' > 0$ because $y' < 0$. Thus, as intended, air resistance is acting in the opposite direction as gravity and slowing down the fall.

Determine the general solution of the DE.

Solution. Substituting $v = y'$, the DE becomes $v' + \rho v = -g$.

This is a linear DE. To solve it, we determine that the integrating factor is $\exp(\int \rho \mathrm{d}t) = e^{\rho t}$.

Multiplying the DE with that factor and integrating, we obtain $e^{\rho t} v = \int -g e^{\rho t} \mathrm{d}t = -\frac{g}{\rho} e^{\rho t} + C$.

Hence, $v(t) = -\frac{g}{\rho} + C e^{-\rho t}$.

Correspondingly, the general solution of the DE is $y(t) = \int v(t) \mathrm{d}t = -\frac{g}{\rho} t - \frac{C}{\rho} e^{-\rho t} + D$.

Comment. Note that $\lim_{t \to \infty} v(t) = -\frac{g}{\rho}$. In other words, the **terminal velocity** is $v_\infty = -\frac{g}{\rho}$.

This is an interesting mathematical consequence of the DE. (And important for the idea behind a parachute!)

Note that, if we know that there is a terminal speed, then we can actually determine its value v_∞ from the DE without solving it by setting $v' = 0$ (because, once the terminal speed is reached, the velocity does not change anymore) in $v' + \rho v = -g$. This gives us $\rho v_\infty = -g$ and, hence, $v_\infty = -g / \rho$ as above.

Let us have another look at Example 6. Note that the DE is a second-order linear differential equation with constant coefficients. Our upcoming goal will be to solve all such equations.

Example 51. Find the general solution to $y'' = y' + 6y$.

Solution. We look for solutions of the form e^{rx}.

Plugging e^{rx} into the DE, we get $r^2 e^{rx} = r e^{rx} + 6e^{rx}$ which simplifies to $r^2 - r - 6 = 0$.

This is called the **characteristic equation**. Its solutions are $r = -2, 3$ (the **characteristic roots**).

This means we found the two solutions $y_1 = e^{-2x}$, $y_2 = e^{3x}$.

The general solution to the DE is $C_1 e^{-2x} + C_2 e^{3x}$.

Comment. In the final step, we used an important principle that is true for linear (!) homogeneous DEs. Namely, if we have solutions $y_1, y_2, \ldots$ then any linear combination $C_1 y_1 + C_2 y_2 + \ldots$ is a solution as well. We will discuss this soon but, for now, check that $C_1 e^{-2x} + C_2 e^{3x}$ is indeed a solution by plugging it into the DE.

Spotlight on the exponential function

Example 52. Solve $y' = ky$ where k is a constant.

Solution. (experience) At this point, we can probably see that $y(x) = e^{kx}$ is a solution.

In fact, the general solution is $y(x) = Ce^{kx}$.

That there cannot be any further solutions follows from the existence and uniqueness theorem (see next example).

Solution. (separation of variables) Alternatively, we can solve the DE using separation of variables.

Express the DE as $\frac{\mathrm{d}y}{\mathrm{d}x} = ky$, then write it as $\frac{1}{y}\mathrm{d}y = k\,\mathrm{d}x$ (note that we just lost the solution $y = 0$).

Integrating gives $\ln|y| = kx + D$, hence $|y| = e^{kx+D}$.

Since the RHS is never zero, $y = \pm e^{kx+D} = Ce^{kx}$ (with $C = \pm e^D$). Finally, note that $C = 0$ corresponds to the singular solution $y = 0$ that we lost. In summary, the general solution is Ce^{kx}.

Example 53. Consider the IVP $y' = ky$, $y(a) = b$. Discuss existence and uniqueness of solutions.

Solution. The IVP is $y' = f(x, y)$ with $f(x, y) = ky$. We compute that $\frac{\partial}{\partial y} f(x, y) = k$.

We observe that both $f(x, y)$ and $\frac{\partial}{\partial y} f(x, y)$ are continuous for all (x, y).

Hence, for any initial conditions, the IVP locally has a unique solution by the existence and uniqueness theorem.

Comment. As a consequence, there can be no other solutions to the DE $y' = ky$ than the ones of the form $y(x) = Ce^{kx}$. Why?! [Assume that $y(x)$ satisfies $y' = ky$ and let (a, b) any value on the graph of y. Then $y(x)$ solves the IVP $y' = ky$, $y(a) = b$; but so does Ce^{kx} with $C = b/e^{ka}$. The uniqueness implies that $y(x) = Ce^{kx}$.]

In particular, we have the following characterization of the exponential function:

e^x is the unique solution to the IVP $y' = y$, $y(0) = 1$.

Comment. Note that, for instance, $\frac{\mathrm{d}}{\mathrm{d}x} 2^x = \ln(2)\, 2^x$. (This follows from $2^x = e^{\ln(2^x)} = e^{x \ln(2)}$.)

Since $\ln = \log_e$, this means that we cannot avoid the natural base $e \approx 2.718$ even if we try to use another base.

Excursion: Euler's identity

Theorem 54. (Euler's identity) $e^{ix} = \cos(x) + i\sin(x)$

Proof. Observe that both sides are the (unique) solution to the IVP $y' = iy$, $y(0) = 1$.
[Check that by computing the derivatives and verifying the initial condition! As we did in class.] $\square$

On lots of T-shirts. In particular, with $x = \pi$, we get $\boxed{e^{\pi i} = -1}$ or $e^{i\pi} + 1 = 0$ (which connects the five fundamental constants).

Example 55. Where do trig identities like $\sin(2x) = 2\cos(x)\sin(x)$ or $\sin^2(x) = \frac{1-\cos(2x)}{2}$ (and infinitely many others!) come from?

Short answer: they all come from the simple exponential law $e^{x+y} = e^x e^y$.

Let us illustrate this in the simple case $(e^x)^2 = e^{2x}$. Observe that

$$\begin{aligned} e^{2ix} &= \cos(2x) + i\sin(2x) \\ e^{ix}e^{ix} &= [\cos(x) + i\sin(x)]^2 = \cos^2(x) - \sin^2(x) + 2i\cos(x)\sin(x). \end{aligned}$$

Comparing imaginary parts (the "stuff with an i"), we conclude that $\sin(2x) = 2\cos(x)\sin(x)$.
Likewise, comparing real parts, we read off $\cos(2x) = \cos^2(x) - \sin^2(x)$.

(Use $\cos^2(x) + \sin^2(x) = 1$ to derive $\sin^2(x) = \frac{1-\cos(2x)}{2}$ from the last equation.)

Challenge. Can you find a triple-angle trig identity for $\cos(3x)$ and $\sin(3x)$ using $(e^x)^3 = e^{3x}$?
Or, use $e^{i(x+y)} = e^{ix}e^{iy}$ to derive $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x+y) = \ldots$

Realize that the complex number $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ corresponds to the point $(\cos(\theta), \sin(\theta))$.
These are precisely the points on the unit circle!

Recall that a point (x, y) can be represented using **polar coordinates** (r, θ), where r is the distance to the origin and θ is the angle with the x-axis.

Then, $x = r\cos\theta$ and $y = r\sin\theta$.

Every complex number z can be written in **polar form** as $z = re^{i\theta}$, with $r = |z|$.

Why? By comparing with the usual polar coordinates ($x = r\cos\theta$ and $y = r\sin\theta$), we can write

$$z = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}.$$

In the final step, we used Euler's identity.

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