Example 126. Solve the IVP $y'' - 3y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$.

Solution. (old style) The characteristic polynomial $D^2 - 3D + 2 = (D-1)(D-2)$ has ("old") roots $1, 2$. The "new" root is -1. Since there is no duplication, there must be a particular solution of the form $y_p(t) = Ae^{-t}$. To determine A, we plug into the DE $y_p'' - 3y_p' + 2y_p = 6Ae^{-t} \overset{!}{=} e^{-t}$ and conclude $A = \frac{1}{6}$.

The general solution thus is $y(t) = \frac{1}{6}e^{-t} + C_1 e^t + C_2 e^{2t}$. We need to find C_1 and C_2 using the initial conditions.

Solving $y(0) = \frac{1}{6} + C_1 + C_2 \overset{!}{=} 0$ and $y'(0) = -\frac{1}{6} + C_1 + 2C_2 \overset{!}{=} 1$, we find $C_2 = \frac{4}{3}$ and $C_1 = -\frac{3}{2}$.

Hence, the unique solution to the IVP is $y(t) = \frac{1}{6}e^{-t} - \frac{3}{2}e^t + \frac{4}{3}e^{2t}$.

Solution. (Laplace style) The differential equation transforms as follows:

$$\begin{aligned}
\mathcal{L}(y''(t)) - 3\mathcal{L}(y'(t)) + 2\mathcal{L}(y(t)) &= \mathcal{L}(e^{-t}) \\
s^2 Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) &= \frac{1}{s+1} \\
(s^2 - 3s + 2)Y(s) &= 1 + \frac{1}{s+1} = \frac{s+2}{s+1} \\
Y(s) &= \frac{s+2}{(s-1)(s-2)(s+1)}
\end{aligned}$$

To find $y(t)$, we use partial fractions to write $Y(s) = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+1}$. We find the coefficients as

$$A = \left.\frac{s+2}{(s-2)(s+1)}\right|_{s=1} = -\frac{3}{2}, \quad B = \left.\frac{s+2}{(s-1)(s+1)}\right|_{s=2} = \frac{4}{3}, \quad C = \left.\frac{s+2}{(s-1)(s-2)}\right|_{s=-1} = \frac{1}{6}.$$

Hence, $y(t) = \mathcal{L}^{-1}\left(\frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s+1}\right) = Ae^t + Be^{2t} + Ce^{-t} = \frac{1}{6}e^{-t} - \frac{3}{2}e^t + \frac{4}{3}e^{2t}$, as above.

Comment. Note the factor $s^2 - 3s + 2$ in front of $Y(s)$ when we transformed the DE. This is the characteristic polynomial. Can you see how the "old" and "new" roots show up in the Laplace transform approach?

Example 127. Consider the IVP $y'' - 3y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$. Determine the Laplace transform of the unique solution.

Solution. We just did that! By transforming the DE, we found that $Y(s) = \frac{s+2}{(s-1)(s-2)(s+1)}$.

Example 128. Consider the IVP $y'' - 3y' + y = 2e^{5t}$, $y(0) = -1$, $y'(0) = 4$. Determine the Laplace transform of the unique solution.

Solution. The DE $y'' - 3y' + y = 2e^{5t}$ transforms into

$$s^2 Y - sy(0) - y'(0) - 3(sY - y(0)) + Y = (s^2 - 3s + 1)Y + (s - 7) = \frac{2}{s-5}$$

Accordingly, $Y(s) = \frac{1}{s^2 - 3s + 1}\left[\frac{2}{s-5} - s + 7\right]$ is the Laplace transform of the unique solution to the IVP.

Comment. The characteristic roots are $(3 \pm \sqrt{5})/2$. As a result, the solution $y(t)$ will be rather unpleasant to write down by hand, with coefficients that are not rational numbers. By contrast, the above Laplace transform can be expressed without irrational numbers.

Comment. Depending on what we intend to do with the solution, we might not even need $y(t)$ but might instead be able to extract what we want from its Laplace transform $Y(s)$.

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We solved the following system in Example 92 using elimination and our method for solving linear DEs with constant coefficients based on characteristic roots.

Example 129. (extra) Solve the system $y_1' = 5y_1 + 4y_2$, $y_2' = 8y_1 + y_2$, $y_1(0) = 0$, $y_2(0) = 1$.

Solution. (using Laplace transforms) $y_1' = 5y_1 + 4y_2$ transforms into $sY_1 - \underbrace{y_1(0)}_{=0} = 5Y_1 + 4Y_2$.

Likewise, $y_2' = 8y_1 + y_2$ transforms into $sY_2 - \underbrace{y_2(0)}_{=1} = 8Y_1 + Y_2$.

The transformed equations are regular equations that we can solve for Y_1 and Y_2.

For instance, by the first equation, $Y_2 = \frac{1}{4}(s-5)Y_1$.

Used in the second equation, we get $\underbrace{-8Y_1 + \frac{1}{4}(s-1)(s-5)Y_1}_{=\frac{1}{4}(s^2-6s-27)=\frac{1}{4}(s+3)(s-9)} = 1$ so that $Y_1 = \frac{4}{(s+3)(s-9)}$.

Hence, the system is solved by $Y_1 = \frac{4}{(s+3)(s-9)}$ and $Y_2 = \frac{1}{4}(s-5)Y_1 = \frac{s-5}{(s+3)(s-9)}$.

As a final step, we need to take the inverse Laplace transform to get $y_1(t) = \mathcal{L}^{-1}(Y_1(s))$ and $y_2(t) = \mathcal{L}^{-1}(Y_2(s))$.

Using partial fractions, $Y_1(s) = \frac{4}{(s+3)(s-9)} = -\frac{1}{3} \cdot \frac{1}{s+3} + \frac{1}{3} \cdot \frac{1}{s-9}$ so that $y_1(t) = -\frac{1}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

Similarly, $Y_2(s) = \frac{s-5}{(s+3)(s-9)} = \frac{2}{3} \cdot \frac{1}{s+3} + \frac{1}{3} \cdot \frac{1}{s-9}$ so that $y_1(t) = \frac{2}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

Solution. (old solution, for comparison) Since $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1$ (from the first eq.), we have $y_2' = \frac{1}{4}y_1'' - \frac{5}{4}y_1'$.

Using these in the second equation, we get $\frac{1}{4}y_1'' - \frac{5}{4}y_1' = 8y_1 + \frac{1}{4}y_1' - \frac{5}{4}y_1$.

Simplified, this is $y_1'' - 6y_1' - 27y_1 = 0$.

This is a homogeneous linear DE with constant coefficients. The characteristic roots are $-3, 9$.

We therefore obtain $y_1 = C_1e^{-3t} + C_2e^{9t}$ as the general solution.

Thus, $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1 = \frac{1}{4}(-3C_1e^{-3t} + 9C_2e^{9t}) - \frac{5}{4}(C_1e^{-3t} + C_2e^{9t}) = -2C_1e^{-3t} + C_2e^{9t}$.

We determine the (unique) values of C_1 and C_2 using the initial conditions:

$y_1(0) = C_1 + C_2 \overset{!}{=} 0$

$y_2(0) = -2C_1 + C_2 \overset{!}{=} 1$

We solve these two equations and find $C_1 = -\frac{1}{3}$ and $C_2 = \frac{1}{3}$.

The unique solution to the IVP therefore is $y_1(t) = -\frac{1}{3}e^{-3t} + \frac{1}{3}e^{9t}$ and $y_2(t) = \frac{2}{3}e^{-3t} + \frac{1}{3}e^{9t}$.

Further entries in the Laplace transform table

We next expand our table of Laplace transforms to the following:

$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
e^{at}	$\frac{1}{s-a}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$
$e^{at}f(t)$	$F(s-a)$
$tf(t)$	$-F'(s)$
$u_a(t)f(t-a)$	$e^{-sa}F(s)$

Example 130.

$$\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-st}e^{at}f(t)\mathrm{d}t = \int_0^\infty e^{-(s-a)t}f(t)\mathrm{d}t = F(s-a)$$

Example 131. We also add the following to our table of Laplace transforms.

$$\mathcal{L}(tf(t)) = \int_0^\infty e^{-st}tf(t)\mathrm{d}t = \int_0^\infty -\frac{\mathrm{d}}{\mathrm{d}s}e^{-st}f(t)\mathrm{d}t = -\frac{\mathrm{d}}{\mathrm{d}s}\int_0^\infty e^{-st}f(t)\mathrm{d}t = -F'(s)$$

In particular,

$$\begin{aligned}\mathcal{L}(t) &= \mathcal{L}(t\cdot 1) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s} = \frac{1}{s^2}\\ \mathcal{L}(t^2) &= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s^2} = \frac{2}{s^3}\\ &\vdots\\ \mathcal{L}(t^n) &= \frac{n!}{s^{n+1}}.\end{aligned}$$

Example 132. Determine the Laplace transform $\mathcal{L}((t-3)e^{2t})$.

Solution. $\mathcal{L}((t-3)e^{2t}) = \mathcal{L}(te^{2t}) - 3\mathcal{L}(e^{2t}) = \frac{1}{(s-2)^2} - \frac{3}{s-2}$

Here, we combined $\mathcal{L}(tf(t)) = -F'(s)$ with $\mathcal{L}(e^{2t}) = \frac{1}{s-2}$ to get $\mathcal{L}(te^{2t}) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s-2} = \frac{1}{(s-2)^2}$.

Example 133. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right)$.

Solution. $\mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right) = e^{3t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = te^{3t}$.

Example 134. (bonus) Solve the IVP $y'' - 3y' + 2y = e^t$, $y(0) = 0$, $y'(0) = 1$.

Solution. (old style, outline) The characteristic polynomial $D^2 - 3D + 2 = (D-1)(D-2)$. Since there is duplication, we have to look for a particular solution of the form $y_p = Ate^t$. To determine A, we need to plug into the DE (we find $A = -1$). Then, the general solution is $y(t) = Ate^t + C_1e^t + C_2e^{2t}$, and the initial conditions determine C_1 and C_2 (we find $C_1 = -2$ and $C_2 = 2$).

Solution. (Laplace style)

$$\begin{aligned}\mathcal{L}(y''(t)) - 3\mathcal{L}(y'(t)) + 2\mathcal{L}(y(t)) &= \mathcal{L}(e^t)\\ s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) &= \frac{1}{s-1}\\ (s^2 - 3s + 2)Y(s) &= 1 + \frac{1}{s-1} = \frac{s}{s-1}\\ Y(s) &= \frac{s}{(s-1)^2(s-2)}\end{aligned}$$

To find $y(t)$, we again use partial fractions. $Y(s) = \frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-2}$ with coefficients (why?!)

$$C = \left.\frac{s}{(s-1)^2}\right|_{s=2} = 2,\quad A = \left.\frac{s}{s-2}\right|_{s=1} = -1,\quad B = \left.\frac{\mathrm{d}}{\mathrm{d}s}\frac{s}{s-2}\right|_{s=1} = \left.\frac{-2}{(s-2)^2}\right|_{s=1} = -2.$$

Finally, $y(t) = \mathcal{L}^{-1}\left(\frac{A}{(s-1)^2} + \frac{B}{s-1} + \frac{C}{s-2}\right) = Ate^t + Be^t + Ce^{2t} = -(t+2)e^t + 2e^{2t}$.

Handling discontinuities with the Laplace transform — bonus

Let $u_a(t) = \begin{cases} 1, & \text{if } t \geqslant a, \\ 0, & \text{if } t < a, \end{cases}$ be the **unit step function**.

Comment. The special case $u_0(t)$ is also known as the **Heaviside function**, after Oliver Heaviside who, among many other things, coined terms like conductance and impedance. Note that $u_a(t) = u_0(t-a)$.

Example 135. $$\mathcal{L}(u_a(t)) = \int_0^\infty e^{-st} u_a(t) \mathrm{d}t = \int_a^\infty e^{-st} \mathrm{d}t = \left[-\frac{e^{-st}}{s}\right]_{t=a}^\infty = \frac{e^{-sa}}{s}.$$

Example 136. Note that $u_a(t) f(t-a)$ is $f(t)$ delayed by a (make a sketch!). We find

$$\mathcal{L}(u_a(t) f(t-a)) = \int_a^\infty e^{-st} f(t-a) \mathrm{d}t = \int_0^\infty e^{-s(\tilde{t}+a)} f(\tilde{t}) \mathrm{d}\tilde{t} = e^{-sa} F(s).$$

Example 137. What is $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+1}\right)$?

Solution. $\frac{1}{s+1}$ is the Laplace transform of e^{-t}. Hence, $\frac{e^{-2s}}{s+1}$ is the Laplace transform of e^{-t} delayed by 2. In other words, $\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s+1}\right) = u_2(t) e^{-(t-2)}$.

The next example illustrates that any piecewise defined function can be written using a single formula involving step functions. This is based on the simple observation that $u_a(t) - u_b(t)$ is a function which is 1 on the interval $[a, b)$ but zero everywhere else.

Example 138. Consider $f(t) = \begin{cases} t^2, & \text{if } 0 \leqslant t \leqslant 1, \\ 1, & \text{if } 1 \leqslant t \leqslant 2, \\ \cos(t-2), & \text{if } t \geqslant 2. \end{cases}$

Then, $f(t) = t^2(u_0(t) - u_1(t)) + 1(u_1(t) - u_2(t)) + \cos(t-2) u_2(t)$.

It is left as an exercise to compute the Laplace transform of $f(t)$ from here. Note that, for instance, to find $\mathcal{L}(t^2 u_1(t))$, we want to use $\mathcal{L}(u_a(t) f(t-a)) = e^{-sa} F(s)$ with $a = 1$ and $f(t-1) = t^2$; then, $f(t) = (t+1)^2 = t^2 + 2t + 1$ has Laplace transform $F(s) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}$, and we combine to get $\mathcal{L}(t^2 u_1(t)) = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$.

Example 139. Solve the IVP $y''+3y'+2y=f(t)$, $y(0)=y'(0)=0$ with $f(t)=\begin{cases} 1, & t\in[3,4],\\ 0, & \text{otherwise.}\end{cases}$

Solution. First, we write $f(t)=u_3(t)-u_4(t)$. We can now take the Laplace transform of the DE to get

$$s^2Y(s)-sy(0)-y'(0)+3(sY(s)-y(0))+2Y(s)=\frac{e^{-3s}}{s}-\frac{e^{-4s}}{s}=(e^{-3s}-e^{-4s})\frac{1}{s}.$$

Using that $s^2+3s+2=(s+1)(s+2)$, we find

$$Y(s)=(e^{-3s}-e^{-4s})\frac{1}{s(s+1)(s+2)}=(e^{-3s}-e^{-4s})\left[\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2}\right],$$

where A, B, C are determined by partial fractions (we compute them below). Taking the inverse Laplace transform of each of the six terms in this product, as in Example 137, we find

$$y(t)=A(u_3(t)-u_4(t))+B(u_3(t)e^{-(t-3)}-u_4(t)e^{-(t-4)})+C(u_3(t)e^{-2(t-3)}-u_4(t)e^{-2(t-4)}).$$

If preferred, we can express this as $y(t)=\begin{cases} 0, & \text{if } t\leqslant 3,\\ A+Be^{-(t-3)}+Ce^{-2(t-3)}, & \text{if } t\in[3,4],\\ B(e^{-(t-3)}-e^{-(t-4)})+C(e^{-2(t-3)}-e^{-2(t-4)}) & \text{if } t\geqslant 4.\end{cases}$

Finally, $A=\frac{1}{(s+1)(s+2)}\Big|_{s=0}=\frac{1}{2}$, $B=\frac{1}{s(s+2)}\Big|_{s=-1}=-1$, $C=\frac{1}{s(s+1)}\Big|_{s=-2}=\frac{1}{2}$.

Comment. Check that these values make $y(t)$ a continuous function (as it should be for physical reasons).