The Laplace transform

Definition 113. The **Laplace transform** of a function $f(t)$, $t \geqslant 0$, is defined as the new function

$$F(s) = \int_0^\infty e^{-st} f(t) \mathrm{d}t.$$

We also write $\mathcal{L}(f(t)) = F(s)$.

Note that, in order for the integral to exist, $f(t)$ should be, say, piecewise continuous and of at most exponential growth. That's true for most of the functions, we are interested in (and so we will not dwell on this issue).

$f(t)$	$F(s)$
e^{at}	$\frac{1}{s-a}$
1	$\frac{1}{s}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$c_1 f_1(t) + c_2 f_2(t)$	$c_1 F_1(s) + c_2 F_2(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - s f(0) - f'(0)$

First entries in the Laplace transform table

In this section, we will discuss and obtain the entries in the table of the most basic Laplace transforms that we compiled after Definition 113.

Example 114. Show that $\mathcal{L}(e^{at}) = \frac{1}{s-a}$.

In particular, in the special case $a = 0$, we have $\mathcal{L}(1) = \frac{1}{s}$.

Solution. $\mathcal{L}(e^{at}) = \int_0^\infty e^{-st} e^{at} \mathrm{d}t = \int_0^\infty e^{(a-s)t} \mathrm{d}t = \left[\frac{1}{a-s} e^{(a-s)t}\right]_{t=0}^\infty = 0 - \frac{1}{a-s} = \frac{1}{s-a}$

Comment. Note that we needed $a - s < 0$ in order for the integral to converge. Hence the Laplace transform has domain $s > a$. (During this introduction, we will not care too much about these technical details.)

In particular. Note that setting $a = 0$ shows that $\mathcal{L}(1) = \frac{1}{s}$.

Example 115. **(linearity)** Show that $\mathcal{L}(c_1 f_1(t) + c_2 f_2(t)) = c_1 F_1(s) + c_2 F_2(s)$.

This means that the Laplace transform is a **linear operator** (like the derivative or the integral).

Solution.

$$\begin{aligned}\mathcal{L}(c_1 f_1(t) + c_2 f_2(t)) &= \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t)) \mathrm{d}t \\ &= c_1 \underbrace{\int_0^\infty e^{-st} f_1(t) \mathrm{d}t}_{F_1(s)} + c_2 \underbrace{\int_0^\infty e^{-st} f_2(t) \mathrm{d}t}_{F_2(x)}\end{aligned}$$

Example 116. **(extra)** Show that $\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$ and $\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2}$.

Solution. By Euler's identity, $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$. Hence, by linearity,

$$\mathcal{L}(e^{i\omega t}) = \mathcal{L}(\cos(\omega t)) + i\mathcal{L}(\sin(\omega t)).$$

On the other hand,

$$\mathcal{L}(e^{i\omega t}) = \frac{1}{s - i\omega} = \frac{s + i\omega}{s^2+\omega^2} = \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}.$$

Matching real and imaginary parts, we find $\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$ and $\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2}$.

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Example 117. Determine $\mathcal{L}(e^{3t} - 7e^{-2t})$.

Solution. $\mathcal{L}(e^{3t} - 7e^{-2t}) = \mathcal{L}(e^{3t}) - 7\mathcal{L}(e^{-2t}) = \frac{1}{s-3} - \frac{7}{s+2}$

Comment. Note that, once we write $\frac{1}{s-3} - \frac{7}{s+2} = -\frac{6s-23}{s^2-s-6}$ it is no longer visibly clear which function we have taken the Laplace transform of. We discuss reversing this process in the next section.

Example 118. (extra) Determine $\mathcal{L}(3\cos(2t) - 5\sin(2t))$.

Solution. $\mathcal{L}(3\cos(2t) - 5\sin(2t)) = 3\mathcal{L}(\cos(2t)) - 5\mathcal{L}(\sin(2t)) = 3\frac{s}{s^2+4} - 5\frac{2}{s^2+4} = \frac{3s-10}{s^2+4}$

Example 119. Show that $\mathcal{L}(f'(t)) = sF(s) - f(0)$.

Solution. Using integration by parts,

$$\mathcal{L}(f'(t)) = \int_0^\infty e^{-st} f'(t) \mathrm{d}t = \left[e^{-st} f(t)\right]_{t=0}^\infty + \int_0^\infty s e^{-st} f(t) \mathrm{d}t = sF(s) - f(0).$$

Higher derivatives. In order to obtain the Laplace transform of higher derivatives, we can iterate. For instance,

$$\mathcal{L}(f''(t)) = s\mathcal{L}(f'(t)) - f'(0) = s[sF(s) - f(0)] - f'(0) = s^2 F(s) - sf(0) - f'(0).$$

The inverse Laplace transform

Theorem 120. (uniqueness of Laplace transforms) If $\mathcal{L}(f_1(t)) = \mathcal{L}(f_2(t))$, then $f_1(t) = f_2(t)$. Hence, we can recover $f(t)$ from $F(s)$. We write $\mathcal{L}^{-1}(F(s)) = f(t)$.

We say that $f(t)$ is the **inverse Laplace transform** of $F(s)$.

Advanced comment. This uniqueness is true for continuous functions f_1, f_2. It is also true for piecewise continuous functions except at those values of t for which there is a discontinuity. (Note that redefining $f(t)$ at a single point, will not change its Laplace transform.)

Example 121. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{5}{s+3}\right)$.

Solution. In other words, if $F(s) = \frac{5}{s+3}$, what is $f(t)$?
$\mathcal{L}^{-1}\left(\frac{5}{s+3}\right) = 5\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = 5e^{-3t}$

Example 122. (extra) Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{3s-7}{s^2+4}\right)$.

Solution. In other words, if $F(s) = \frac{3s-7}{s^2+4}$, what is $f(t)$?
$F(s) = 3\frac{s}{s^2+2^2} - \frac{7}{2}\frac{2}{s^2+2^2}$. Hence, $f(t) = 3\cos(2t) - \frac{7}{2}\sin(2t)$.

Example 123. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right)$.

Solution. Note that $s^2 - s - 6 = (s-3)(s+2)$. We use **partial fractions** to write $-\frac{6s-23}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$. We find the coefficients as

$$A = -\frac{6s-23}{s+2}\bigg|_{s=3} = 1, \quad B = -\frac{6s-23}{s-3}\bigg|_{s=-2} = -7.$$

Hence $\mathcal{L}^{-1}\left(-\frac{6s-23}{s^2-s-6}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-3} - \frac{7}{s+2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s-3}\right) - 7\mathcal{L}^{-1}\left(\frac{7}{s+2}\right) = e^{3t} - 7e^{-2t}$.

Review. In order to find A, we multiply $-\frac{6s-23}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$ by $s-3$ to get $-\frac{6s-23}{s+2} = A + \frac{B(s-3)}{s+2}$. We then set $s=3$ to find A as above.

Comment. Compare with Example 117 where we considered the same functions.

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Example 124. Determine the inverse Laplace transform $\mathcal{L}^{-1}\left(\frac{s+13}{s^2-s-2}\right)$.

Solution. Note that $s^2-s-2=(s-2)(s+1)$. We use partial fractions to write $\frac{s+13}{(s-2)(s+1)}=\frac{A}{s-2}+\frac{B}{s+1}$. We find the coefficients as

$$A=\left.\frac{s+13}{s+1}\right|_{s=2}=5, \quad B=\left.\frac{s+13}{s-2}\right|_{s=-1}=-4.$$

Hence $\mathcal{L}^{-1}\left(\frac{s+13}{s^2-s-2}\right)=\mathcal{L}^{-1}\left(\frac{5}{s+1}-\frac{4}{s-2}\right)=5e^t-4e^{2t}$.

Solving simple DEs using the Laplace transform

In the following examples, we write $Y(s)$ for the Laplace transform of $y(t)$.

Example 125. Solve the (very simple) IVP $y'(t)-2y(t)=0$, $y(0)=7$.

At this point, you might be able to "see" right away that the unique solution is $y(t)=7e^{2t}$.

Solution. (old style) The characteristic root is 2, so that the general solution is $y(t)=Ce^{2t}$. Using the initial condition, we find that $C=7$, so that $y(t)=7e^{2t}$.

Solution. (Laplace style) $y'-2y=0$ transforms into

$$\mathcal{L}(y'(t)-2y(t))=\mathcal{L}(y'(t))-2\mathcal{L}(y(t))=sY(s)-y(0)-2Y(s)=(s-2)Y(s)-7=0.$$

This is an algebraic equation for $Y(s)$. It follows that $Y(s)=\frac{7}{s-2}$. By inverting the Laplace transform, we conclude that $y(t)=7e^{2t}$.

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