Adding external forces and the phenomenon of resonance

The motion of a mass on a spring, with an external force $f(t)$ taken into account, can be modeled by the DE

$$y'' + cy = f(t).$$

Example 106. Describe the solutions of $y'' + 4y = \cos(\lambda t)$.

Solution. The "old" roots are $\pm 2i$ so that 2 is the **natural frequency** (the frequency at which the system would oscillate in the absence of external forces; mathematically, this reflects the fact that the general solution to the corresponding homogeneous DE is $A\cos(2t) + B\sin(2t)$, which has frequency $\omega = 2$).

The "new" roots are $\pm\lambda i$ where λ is the **external frequency**.

Case 1: $\lambda \neq 2$. Then there is a particular solution of the form $y_p = A\cos(\lambda t) + B\sin(\lambda t)$. To determine the unique values of A, B, we plug into the DE:

$$y_p'' + 4y_p = (4 - \lambda^2)A\cos(\lambda t) + (4 - \lambda^2)B\sin(\lambda t) \stackrel{!}{=} \cos(\lambda t)$$

We conclude that $(4 - \lambda^2)A = 1$ and $(4 - \lambda^2)B = 0$. Solving these, we find $A = 1/(4 - \lambda^2)$ and $B = 0$.

Thus, the general solution is of the form $y = \frac{1}{4 - \lambda^2}\cos(\lambda t) + C_1\cos(2t) + C_2\sin(2t)$.

Case 2: $\lambda = 2$. Now, there is a particular solution of the form $y_p = At\cos(2t) + Bt\sin(2t)$. To determine the unique values of A, B, we again plug into the DE (which is more work this time):

$$y_p'' + 4y_p \stackrel{\text{work}}{=} 4B\cos(2t) - 4A\sin(2t) \stackrel{!}{=} \cos(2t)$$

We conclude that $4B = 1$ and $-4A = 0$. Solving these, we find $A = 0$ and $B = 1/4$.

Thus, the general solution is of the form $y = \frac{1}{4}t\sin(2t) + C_1\cos(2t) + C_2\sin(2t)$.

Note that the amplitude in y_p increases without bound (so that the same is true for the general solution).

This phenomenon is called **resonance**; it occurs if an external frequency matches a natural frequency.

> If an external frequency matches a natural frequency, then **resonance** occurs.
>
> In that case, we obtain amplitudes that grow without bound.

Resonance (or anything close to it) is very important for practical purposes because large amplitudes can be very destructive: singing to shatter glass, earth quake waves and buildings, marching soldiers on bridges, ...

Comment. Mathematically speaking, the "old" and "new" roots overlap in an inhomogeneous linear DE. In that case, the solutions acquire a factor of the variable t (or x) which changes the nature of the solutions.

Example 107. Consider $y'' + 9y = 10\cos(2\lambda t)$. For what value of λ does resonance occur?

Solution. The natural frequency is 3. The external frequency is 2λ. Hence, resonance occurs when $\lambda = \frac{3}{2}$.

Example 108. The motion of a mass on a spring under an external force is described by $5y'' + 2y = -2\sin(3\lambda t)$. For which value of λ does resonance occur?

Solution. The natural frequency is $\sqrt{\frac{2}{5}}$. The external frequency is 3λ. Hence, resonance occurs when $\lambda = \frac{1}{3}\sqrt{\frac{2}{5}}$.

Example 109. The motion of a mass on a spring under an external force is described by $3y'' + ry = \cos(t/2)$. For which value of $r > 0$ does resonance occur?

Solution. The natural frequency is $\sqrt{\frac{r}{3}}$. The external frequency is $\frac{1}{2}$. Hence, resonance occurs when $\sqrt{\frac{r}{3}} = \frac{1}{2}$. This happens if $r = 3 \cdot \left(\frac{1}{2}\right)^2 = \frac{3}{4}$.

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External forces plus damping (extra material)

Example 110. Find the general solution of $2y'' + 2y' + y = 10\sin(t)$.

Solution. The "old" roots are $\frac{1}{4}(-2 \pm \sqrt{4-8}) = -\frac{1}{2} \pm \frac{1}{2}i$.

Accordingly, the system without external force is underdamped. (Make sure that this is clear to you!)

Proceeding as usual, we find $y_p = -4\cos(t) - 2\sin(t) = \sqrt{20}(\cos(t-\alpha))$ with $\alpha = \tan^{-1}(1/2) + \pi \approx 3.605$.

Here, we used that $(-4,-2) = \sqrt{20}(\cos\alpha, \sin\alpha)$.

Hence, the general solution is $y(t) = \underbrace{\sqrt{20}\cos(t-\alpha)}_{y_{\mathrm{sp}}} + \underbrace{e^{-t/2}\left(C_1\cos\left(\frac{t}{2}\right) + C_2\sin\left(\frac{t}{2}\right)\right)}_{y_{\mathrm{tr}} \to 0 \text{ as } t \to \infty}$.

Observe how $y = y_{\mathrm{tr}} + y_{\mathrm{sp}}$ splits into **transient** motion y_{tr} and **steady periodic** oscillations y_{sp}.

Example 111. Find the steady periodic solution to $y'' + 2y' + 5y = \cos(\lambda t)$. What is the amplitude of the steady periodic oscillations? For which ω is the amplitude maximal?

Solution. The "old" roots are $-1 \pm 2i$.

[Not really needed, because positive damping prevents duplication; can you see it?]

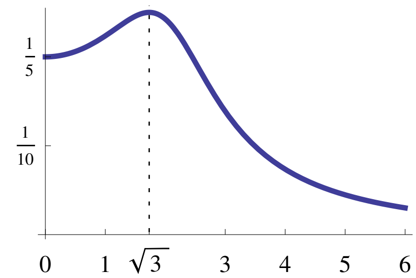
Hence, $y_{\mathrm{sp}} = A\cos(\lambda t) + B\sin(\lambda t)$ and to find A, B we need to plug into the DE.

Doing so, we find $A = \frac{5-\lambda^2}{(5-\lambda^2)^2 + 4\lambda^2}$, $B = \frac{2\lambda}{(5-\lambda^2)^2 + 4\lambda^2}$.

Thus, the amplitude of y_{sp} is $r(\lambda) = \sqrt{A^2 + B^2} = \frac{1}{\sqrt{(5-\lambda^2)^2 + 4\lambda^2}}$.

The function $r(\lambda)$ is sketched to the right. It has a maximum at $\lambda = \sqrt{3}$ at which the amplitude is unusually large (well, here it is not very pronounced). We say that **practical resonance** occurs for $\lambda = \sqrt{3}$.

[For comparison, without damping, (pure) resonance occurs for $\lambda = \sqrt{5}$.]



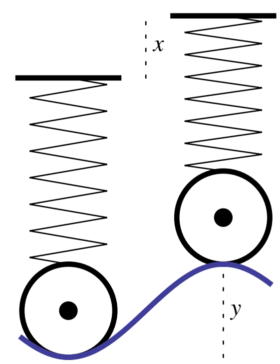
Example 112. A car is going at constant speed v on a washboard road surface ("bumpy road") with height profile $y(s) = a\cos\left(\frac{2\pi s}{L}\right)$. Assume that the car oscillates vertically as if on a spring (no dashpot). Describe the resulting oscillations.

Solution. With x as in the sketch, the spring is stretched by $x - y$. Hence, by Hooke's and Newton's laws, its motion is described by $mx'' = -k(x-y)$.

At constant speed, $s = vt$ and we obtain the DE $mx'' + kx = ky = ka\cos\left(\frac{2\pi vt}{L}\right)$, which is inhomogeneous linear with constant coefficients. Let's solve it.

"Old" roots: $\pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$. $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency.

"New" roots: $i\frac{2\pi v}{L} = \pm i\omega$. $\omega = \frac{2\pi v}{L}$ is the external frequency.



The first "car" is assumed to be in equilibrium.

Case 1: $\omega \neq \omega_0$. Then a particular solution is $x_p = b_1\cos(\omega t) + b_2\sin(\omega t) = A\cos(\omega t - \alpha)$ for unique values of b_1, b_2 (which we do not compute here). The general solution is of the form $x = x_p + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$.

Case 2: $\omega = \omega_0$. Then a particular solution is $x_p = t[b_1\cos(\omega t) + b_2\sin(\omega t)] = At\cos(\omega t - \alpha)$ for unique values of b_1, b_2 (which we do not compute). Note that the amplitude in x_p increases without bound; the same is true for the general solution $x = x_p + C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$. This phenomenon is called resonance; it occurs if an external frequency matches a natural frequency.