Example 92.

(a) Determine the general solution to $y_1' = 5y_1 + 4y_2$, $y_2' = 8y_1 + y_2$.

Comment. In matrix form, with $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, this is $\boldsymbol{y}' = \begin{bmatrix} 5 & 4 \\ 8 & 1 \end{bmatrix}\boldsymbol{y}$.

(b) Solve the IVP $y_1' = 5y_1 + 4y_2$, $y_2' = 8y_1 + y_2$, $y_1(0) = 0$, $y_2(0) = 1$.

(c) Determine a particular solution to $y_1' = 5y_1 + 4y_2 + e^{2x}$, $y_2' = 8y_1 + y_2$.

(d) Determine the general solution to $y_1' = 5y_1 + 4y_2 + e^{2x}$, $y_2' = 8y_1 + y_2$.

Solution.

(a) Since $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1$ (from the first equation), we have $y_2' = \frac{1}{4}y_1'' - \frac{5}{4}y_1'$.
Using these in the second equation, we get $\frac{1}{4}y_1'' - \frac{5}{4}y_1' = 8y_1 + \frac{1}{4}y_1' - \frac{5}{4}y_1$.
Simplified, this is $y_1'' - 6y_1' - 27y_1 = 0$.
This is a homogeneous linear DE with constant coefficients. The characteristic roots are $-3, 9$.
We therefore obtain $y_1 = C_1e^{-3x} + C_2e^{9x}$ as the general solution.
Thus, $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1 = \frac{1}{4}(-3C_1e^{-3x} + 9C_2e^{9x}) - \frac{5}{4}(C_1e^{-3x} + C_2e^{9x}) = -2C_1e^{-3x} + C_2e^{9x}$.

(b) We already have the general solutions y_1, y_2 to the two DEs. We need to determine the (unique) values of C_1 and C_2 to match the initial conditions:
$y_1(0) = C_1 + C_2 \stackrel{!}{=} 0$
$y_2(0) = -2C_1 + C_2 \stackrel{!}{=} 1$
We solve these two equations and find $C_1 = -\frac{1}{3}$ and $C_2 = \frac{1}{3}$.
The unique solution to the IVP therefore is $y_1 = -\frac{1}{3}e^{-3x} + \frac{1}{3}e^{9x}$ and $y_2 = \frac{2}{3}e^{-3x} + \frac{1}{3}e^{9x}$.

(c) We proceed as in the first part to write $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1 - \frac{1}{4}e^{2x}$.
Using this in the second equation, we get $\frac{1}{4}y_1'' - \frac{5}{4}y_1' - \frac{1}{2}e^{2x} = 8y_1 + \frac{1}{4}y_1' - \frac{5}{4}y_1 - \frac{1}{4}e^{2x}$.
Simplified, this is $y_1'' - 6y_1' - 27y_1 = e^{2x}$.
This is an inhomogeneous linear DE with constant coefficients. Since the "old" roots are $-3, 9$, while the "new" root is 2, there must a particular solution of the form $y_1 = Ce^{2x}$. Plugging this y_1 into the DE, we get $y_1'' - 6y_1' - 27y_1 = (4 - 6\cdot 2 - 27)Ce^{2x} = -35Ce^{2x} \stackrel{!}{=} e^{2x}$. Hence, $C = -\frac{1}{35}$.
Corresponding to $y_1 = -\frac{1}{35}e^{2x}$ we get $y_2 = \frac{1}{4}y_1' - \frac{5}{4}y_1 = -\frac{1}{35}\left(\frac{1}{4}\cdot 2 - \frac{5}{4}\right)e^{2x} = \frac{3}{35}e^{2x}$.

(d) The general solution is the particular solution (previous part) plus the general solution to the homogeneous system (first part):
$y_1 = -\frac{1}{35}e^{2x} + C_1e^{-3x} + C_2e^{9x}$
$y_2 = \frac{3}{35}e^{2x} - 2C_1e^{-3x} + C_2e^{9x}$

Higher-order linear DEs as first-order systems

Example 93. Write the (second-order) differential equation $y'' = 2y' + 5y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$ and $y_2 = y'$. Then $y'' = 2y' + 5y$ becomes $y_2' = 2y_2 + 5y_1$.

Therefore, $y'' = 2y' + 5y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = 5y_1 + 2y_2 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 \\ 5 & 2 \end{bmatrix}\boldsymbol{y}$.

Comment. This illustrates why we might care about systems of DEs, even if we work with only one function.

Armin Straub
straub@southalabama.edu

Example 94. Write the (third-order) differential equation $y''' = 3y'' - 2y' + 4y$ as a system of (first-order) differential equations.

Solution. Write $y_1 = y$, $y_2 = y'$ and $y_3 = y''$.

Then, $y''' = 3y'' - 2y' + 4y$ translates into the first-order system $\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = 4y_1 - 2y_2 + 3y_3 \end{cases}$.

In matrix form, this is $\boldsymbol{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -2 & 3 \end{bmatrix} \boldsymbol{y}$.

Example 95. Consider the following system of (second-order) initial value problems:

$$\begin{array}{l} y_1'' = 2y_1' - 3y_2' + 7y_2 \\ y_2'' = 4y_1' + y_2' - 5y_1 \end{array} \qquad y_1(0) = 2,\ y_1'(0) = 3,\ y_2(0) = -1,\ y_2'(0) = 1$$

Write it as a first-order initial value problem in the form $\boldsymbol{y}' = M\boldsymbol{y}$, $\boldsymbol{y}(0) = \boldsymbol{y}_0$.

Solution. Introduce $y_3 = y_1'$ and $y_4 = y_2'$. Then, the given system translates into

$$\boldsymbol{y}' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 7 & 2 & -3 \\ -5 & 0 & 4 & 1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \end{bmatrix}.$$

Extra: Two more applications of systems of DEs

Example 96. (military strategy) Lanchester's equations model two opposing forces during "aimed fire" battle.

Let $x(t)$ and $y(t)$ describe the number of troops on each side. Then Lanchester (during World War I) assumed that the rates $-x'(t)$ and $-y'(t)$, at which soldiers are put out of action, are proportional to the number of opposing forces. That is:

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} -\alpha y(t) \\ -\beta x(t) \end{bmatrix}, \quad \text{or, in matrix form: } \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -\alpha \\ -\beta & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The proportionality constants $\alpha, \beta > 0$ indicate the strength of the forces ("fighting effectiveness coefficients").

These are simple linear DEs with constant coefficients, which we have learned how to solve.

Comment. The "aimed fire" means that all combatants are engaged, as is common in modern combat with long-range weapons. This is rather different than ancient combat where soldiers were engaging one opponent at a time.

For more details, see: `https://en.wikipedia.org/wiki/Lanchester%27s_laws`

Armin Straub
straub@southalabama.edu

Example 97. (epidemiology) Let us indicate the popular SIR model for short outbreaks of diseases among a population of constant size N.

In a SIR model, the population is compartmentalized into $S(t)$ susceptible, $I(t)$ infected and $R(t)$ recovered (or resistant) individuals ($N = S(t) + I(t) + R(t)$). In the Kermack-McKendrick model, the outbreak of a disease is modeled by

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I,$$

with γ modeling the recovery rate and β the infection rate. Note that this is a non-linear system of differential equations. For more details and many variations used in epidemiology, see:

https://en.wikipedia.org/wiki/Compartmental_models_in_epidemiology

Comment. The following variation

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \gamma I R, \quad \frac{\mathrm{d}S}{\mathrm{d}t} = -\beta S I, \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \beta S I - \gamma I R,$$

which assumes "infectious recovery", was used in 2014 to predict that facebook might lose 80% of its users by 2017. It is that claim, not mathematics (or even the modeling), which attracted a lot of media attention.

http://blogs.wsj.com/digits/2014/01/22/controversial-paper-predicts-facebook-decline/

A closer look at second-order linear DEs

Application: motion of a mass on a spring

Example 98. The motion of a mass m attached to a spring is described by

$$m y'' + k y = 0$$

where y is the displacement from the equilibrium position and $k > 0$ is the spring constant.

Why? This follows from Hooke's law $F = -k y$ combined with Newton's second law $F = m a = m y''$. (Note that the minus sign is needed because the force on the mass is in direction opposite to the displacement.)

Comment. By measuring y as the displacement from equilibrium, it doesn't matter whether the mass is attached horizontally or vertically (gravity is taken into account by the extra stretch in the spring due to the mass).

Solving this DE, we find that the general solution is

$$y(t) = A\cos(\omega t) + B\sin(\omega t)$$

where $\omega = \sqrt{k / m}$ (note that the characteristic roots are $\pm i\sqrt{\frac{k}{m}}$). We observe that:

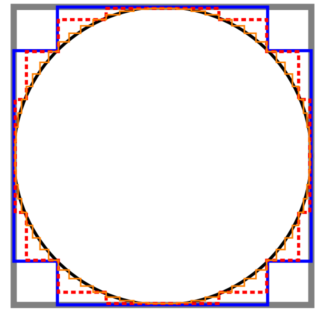
- The motion $y(t)$ is periodic with **period** $2\pi / \omega$.

  This follows from the fact that both $\cos(t)$ and $\sin(t)$ have period 2π.

- The **amplitude** of the motion $y(t)$ is $\sqrt{A^2 + B^2}$.

  This follows from the fact that $y(t) = A\cos(\omega t) + B\sin(\omega t) = r\cos(\omega t - \alpha)$ where (r, α) are the **polar coordinates** for (A, B). In particular, the amplitude is $r = \sqrt{A^2 + B^2}$.
  Can you explain the reason for being able to write $y(t)$ as $r\cos(\omega t - \alpha)$ using DEs? We will do so next time.

(A Halloween scare!) π is the perimeter of a circle enclosed in a square with edge length 1. The perimeter of the square is 4, which approximates π. To get a better approximation, we "fold" the vertices of the square towards the circle (and get the blue polygon). This construction can be repeated for even better approximations and, in the limit, our shape will converge to the true circle. At each step, the perimeter is 4, so we conclude that $\pi = 4$, contrary to popular belief.



Can you pin-point the fallacy in this argument?

(We are not doing something completely silly! For instance, the areas of our approximations do converge to $\pi/4$, the area of the circle.)

The "solution" is below...

($\pi = 4$, "solution")

We are constructing curves c_n with the property that $c_n \to c$ where c is the circle. This convergence can be understood, for instance, in the same sense $\|c_n - c\| \to 0$ with the norm measuring the maximum distance between the two curves.

Since $c_n \to c$ we then wanted to conclude that $\text{perimeter}(c_n) \to \text{perimeter}(c)$, leading to $4 \to \pi$.

However, in order to conclude from $x_n \to x$ that $f(x_n) \to f(x)$ we need that f is continuous (at x)!!

The "function" perimeter, however, is not continuous. In words, this means that (as we see in this example) curves can be arbitrarily close, yet have very different arc length.

We can dig a little deeper: as we learned in Calculus II, the arc length of a function $y = f_n(x)$ for $x \in [a, b]$ is

$$\int_a^b \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \int_a^b \sqrt{1 + f_n'(x)^2}\mathrm{d}x.$$

Observe that this involves $f_n'(x)$. Try to see why the operator D that sends f to f' is not continuous with respect to the distance induced by the norm

$$\|f\| = \left(\int_a^b f(x)^2 \mathrm{d}x\right)^{1/2}.$$

In words, two functions f and g can be arbitrarily close, yet have very different derivatives f' and g'.

That's a huge issue in **functional analysis**, which is the generalization of linear algebra to infinite dimensional spaces (like the space of all differentiable functions). The linear operators ("matrices") on these spaces frequently fail to be continuous.

Armin Straub
straub@southalabama.edu