

**Example 83.** Determine a particular solution of  $y'' + 4y' + 4y = 2e^{3x} - 5e^{-2x}$ .

**Solution.** Write the DE as  $Ly = 2e^{3x} - 5e^{-2x}$  where  $L = D^2 + 4D + 4$ . Instead of starting all over, recall that in Example 81 we found that  $y_1 = \frac{1}{25}e^{3x}$  satisfies  $Ly_1 = e^{3x}$ . Also, in Example 82 we found that  $y_2 = \frac{7}{2}x^2e^{-2x}$  satisfies  $Ly_2 = 7e^{-2x}$ .

By linearity, it follows that  $L(Ay_1 + By_2) = ALy_1 + BLy_2 = Ae^{3x} + 7Be^{-2x}$ .

To get a particular solution  $y_p$  of our DE, we need  $A = 2$  and  $7B = -5$ .

Hence,  $y_p = 2y_1 - \frac{5}{7}y_2 = \frac{2}{25}e^{3x} - \frac{5}{7}x^2e^{-2x}$ .

**Example 84. (homework)** Determine the general solution of  $y'' - 2y' + y = 5\sin(3x)$ .

**Solution.** Since  $D^2 - 2D + 1 = (D - 1)^2$ , the "old" roots are 1, 1. The "new" roots are  $\pm 3i$ . Hence, there has to be a particular solution of the form  $y_p = A\cos(3x) + B\sin(3x)$ .

To find the values of  $A$  and  $B$ , we plug into the DE.

$$y_p' = -3A\sin(3x) + 3B\cos(3x)$$

$$y_p'' = -9A\cos(3x) - 9B\sin(3x)$$

$$y_p'' - 2y_p' + y_p = (-8A - 6B)\cos(3x) + (6A - 8B)\sin(3x) \stackrel{!}{=} 5\sin(3x)$$

Equating the coefficients of  $\cos(x)$ ,  $\sin(x)$ , we obtain the two equations  $-8A - 6B = 0$  and  $6A - 8B = 5$ .

Solving these, we find  $A = \frac{3}{10}$ ,  $B = -\frac{2}{5}$ . Accordingly, a particular solution is  $y_p = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x)$ .

The general solution is  $y(x) = \frac{3}{10}\cos(3x) - \frac{2}{5}\sin(3x) + (C_1 + C_2x)e^x$ .

**Example 85. (homework)** What is the shape of a particular solution of  $y'' + 4y' + 4y = x\cos(x)$ ?

**Solution.** The "old" roots are  $-2, -2$ . The "new" roots are  $\pm i, \pm i$ . Hence, there has to be a particular solution of the form  $y_p = (C_1 + C_2x)\cos(x) + (C_3 + C_4x)\sin(x)$ .

**Continuing to find a particular solution.** To find the value of the  $C_j$ 's, we plug into the DE.

$$y_p' = (C_2 + C_3 + C_4x)\cos(x) + (C_4 - C_1 - C_2x)\sin(x)$$

$$y_p'' = (2C_4 - C_1 - C_2x)\cos(x) + (-2C_2 - C_3 - C_4x)\sin(x)$$

$$y_p'' + 4y_p' + 4y_p = (3C_1 + 4C_2 + 4C_3 + 2C_4 + (3C_2 + 4C_4)x)\cos(x) \\ + (-4C_1 - 2C_2 + 3C_3 + 4C_4 + (-4C_2 + 3C_4)x)\sin(x) \stackrel{!}{=} x\cos(x).$$

Equating the coefficients of  $\cos(x)$ ,  $x\cos(x)$ ,  $\sin(x)$ ,  $x\sin(x)$ , we get the equations  $3C_1 + 4C_2 + 4C_3 + 2C_4 = 0$ ,  $3C_2 + 4C_4 = 1$ ,  $-4C_1 - 2C_2 + 3C_3 + 4C_4 = 0$ ,  $-4C_2 + 3C_4 = 0$ .

Solving (this is tedious!), we find  $C_1 = -\frac{4}{125}$ ,  $C_2 = \frac{3}{25}$ ,  $C_3 = -\frac{22}{125}$ ,  $C_4 = \frac{4}{25}$ .

Hence,  $y_p = \left(-\frac{4}{125} + \frac{3}{25}x\right)\cos(x) + \left(-\frac{22}{125} + \frac{4}{25}x\right)\sin(x)$ .

**Example 86. (homework)** What is the shape of a particular solution of  $y'' + 4y' + 4y = 4e^{3x}\sin(2x) - x\sin(x)$ .

**Solution.** The "old" roots are  $-2, -2$ . The "new" roots are  $3 \pm 2i, \pm i, \pm i$ .

Hence, there has to be a particular solution of the form

$$y_p = C_1e^{3x}\cos(2x) + C_2e^{3x}\sin(2x) + (C_3 + C_4x)\cos(x) + (C_5 + C_6x)\sin(x).$$

**Continuing to find a particular solution.** To find the values of  $C_1, \dots, C_6$ , we plug into the DE. But this final step is so boring that we don't go through it here. Computers (currently?) cannot afford to be as selective; mine obediently calculated:  $y_p = -\frac{4}{841}e^{3x}(20\cos(2x) - 21\sin(2x)) + \frac{1}{125}((-22 + 20x)\cos(x) + (4 - 15x)\sin(x))$

## A more general method for finding particular solutions: variation of parameters

**Review.** To find the general solution of an inhomogeneous linear DE  $Ly = Q(x)$ , we only need to find a single **particular solution**  $y_p$ . Then the general solution is  $y_p + y_h$ , where  $y_h$  is the general solution of  $Ly = 0$ .

**Theorem 87. (variation of parameters)** A particular solution to the inhomogeneous second-order linear DE  $Ly = y'' + P_1(x)y' + P_0(x)y = Q(x)$  is given by:

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x), \quad u_1(x) = - \int \frac{y_2(x)Q(x)}{W(x)} dx, \quad u_2(x) = \int \frac{y_1(x)Q(x)}{W(x)} dx,$$

where  $y_1, y_2$  are independent solutions of  $Ly = 0$  and  $W = y_1y_2' - y_1'y_2$  is their Wronskian.

**Comment.** We obtain the general solution if we consider all possible constants of integration in the formula for  $y_p$ .

**Proof.** Let us look for a particular solution of the form  $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$ .

This “ansatz” is called **variation of constants/parameters**. We plug into the DE to determine conditions on  $u_1, u_2$  so that  $y_p$  is a solution. The DE will give us one condition and (since there are two unknowns), it is reasonable to expect that we can impose a second condition (labelled below as “our wish”) to make our life simpler.

We compute  $y_p' = \underbrace{u_1'y_1 + u_2'y_2}_{=0 \text{ (our wish)}} + u_1y_1' + u_2y_2'$  and, thus,  $y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''$ .

[“Our wish” was chosen so that  $y_p''$  would only involve first derivatives of  $u_1$  and  $u_2$ .]

Therefore, plugging into the DE results in

$$Ly_p = \underbrace{u_1'y_1' + u_2'y_2'}_{=0} + \underbrace{u_1y_1'' + u_2y_2'' + P_1(x)(u_1y_1' + u_2y_2') + P_0(x)(u_1y_1 + u_2y_2)}_{=u_1Ly_1 + u_2Ly_2 = 0} \stackrel{!}{=} Q(x).$$

We conclude that  $y_p$  solves the DE if the following two conditions (the first is “our wish”) are satisfied:

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0, \\ u_1'y_1' + u_2'y_2' &= Q(x). \end{aligned}$$

These are linear equations in  $u_1'$  and  $u_2'$ . Solving gives  $u_1' = \frac{-y_2 Q(x)}{y_1y_2' - y_1'y_2}$  and  $u_2' = \frac{y_1 Q(x)}{y_1y_2' - y_1'y_2}$ , and it only remains to integrate. □

**Comment.** In matrix-vector form, the equations are  $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ Q(x) \end{bmatrix}$ .

Our solution then follows from multiplying  $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}^{-1} = \frac{1}{y_1y_2' - y_1'y_2} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}$  with  $\begin{bmatrix} 0 \\ Q(x) \end{bmatrix}$ .

**Advanced comment.**  $W = y_1y_2' - y_1'y_2$  is called the **Wronskian** of  $y_1$  and  $y_2$ . In general, given a linear homogeneous DE of order  $n$  with solutions  $y_1, \dots, y_n$ , the Wronskian of  $y_1, \dots, y_n$  is the determinant of the matrix where each column consists of the derivatives of one of the  $y_i$ . One useful property of the Wronskian is that it is nonzero if and only if the  $y_1, \dots, y_n$  are linearly independent and therefore generate the general solution.

**Example 88.** Determine the general solution of  $y'' - 2y' + y = \frac{e^x}{x}$ .

**Solution.** This DE is of the form  $Ly = Q(x)$  with  $L = D^2 - 2D + 1$  and  $Q(x) = \frac{e^x}{x}$ .

Since  $L = (D - 1)^2$ , the homogeneous DE has the two solutions  $y_1 = e^x$ ,  $y_2 = xe^x$ .

The corresponding Wronskian is  $W = y_1y_2' - y_1'y_2 = e^{2x}$ .

By variation of parameters (Theorem 87), we find that a particular solution is

$$y_p = -y_1 \int \frac{y_2 Q}{W} dx + y_2 \int \frac{y_1 Q}{W} dx = -e^x \int 1 dx + xe^x \int \frac{1}{x} dx = xe^x[\ln|x| - 1].$$

The general solution therefore is  $xe^x[\ln|x| - 1] + (C_1 + C_2x)e^x$ .

**Comment.** Adding constants of integration in the formula for  $y_p$ , we get  $-e^x(x + D_1) + xe^x(\ln|x| + D_2)$ , which is the general solution. Any choice of constants suffices to give us a particular solution.