Example 73. (review) Solve the IVP $y''' = 8y'' - 16y'$ with $y(0) = 1$, $y'(0) = 4$, $y''(0) = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - 8D^2 + 16D = D(D-4)^2$ has roots $0, 4, 4$.

By Theorem 64, the general solution is $y(x) = C_1 + (C_2 + C_3 x)e^{4x}$.

Using $y'(x) = (4C_2 + C_3 + 4C_3 x)e^{4x}$ and $y''(x) = 4(4C_2 + 2C_3 + 4C_3 x)e^{4x}$, the initial conditions result in the equations $C_1 + C_2 = 1$, $4C_2 + C_3 = 4$, $16C_2 + 8C_3 = 0$.

Solving these (start with the last two equations) we find $C_1 = -1$, $C_2 = 2$, $C_3 = -4$.

Hence the unique solution to the IVP is $y(x) = -1 + (2 - 4x)e^{4x}$.

Important comment. Check (as we did in class) that $y(x)$ indeed solves the IVP.

Example 74. (review) Determine the general solution of $y''' - y'' - 5y' - 3y = 0$.

Solution. The characteristic polynomial $p(D) = D^3 - D^2 - 5D - 3 = (D-3)(D+1)^2$ has roots $3, -1, -1$.

Hence, the general solution is $y(x) = C_1 e^{3x} + (C_2 + C_3 x)e^{-x}$.

Example 75. (review) Find the general solution of $y^{(7)} + 8y^{(6)} + 42y^{(5)} + 104y^{(4)} + 169y''' = 0$.

Use the fact that $-2 + 3i$ is a repeated characteristic root.

Solution. The characteristic polynomial $p(D) = D^3(D^2 + 4D + 13)^2$ has roots $0, 0, 0, -2 \pm 3i, -2 \pm 3i$.

[Since $-2 + 3i$ is a root so must be $-2 - 3i$. Repeating them once, together with $0, 0, 0$ results in 7 roots.]

Hence, the general solution is $(A + Bx + Cx^2) + (D + Ex)\, e^{-2x}\cos(3x) + (F + Gx)e^{-2x}\sin(3x)$.

Inhomogeneous linear DEs

Review. A linear DE of order n is of the form

$$y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \ldots + P_1(x) y' + P_0(x) y = Q(x).$$

- In terms of $D = \frac{\mathrm{d}}{\mathrm{d}x}$, the DE becomes: $Ly = f(x)$ with $L = D^n + P_{n-1}(x) D^{n-1} + \ldots + P_1(x) D + P_0(x)$.
- L is called a **linear differential operator**.
  - $L(c_1 y_1 + c_1 y_2) = c_1 L y_1 + c_2 L y_2$ **(linearity)**
    Comment. If you are familiar with linear algebra, think of L replaced with a matrix A and y_1, y_2 replaced with vectors $\boldsymbol{v}_1, \boldsymbol{v}_2$. In that case, the same linearity property holds.
- The inclusion of the $Q(x)$ term makes $Ly = Q(x)$ an **inhomogeneous** linear DE.
- $Ly = 0$ is the corresponding **homogeneous** DE.
  - If y_1 and y_2 are solutions to the homogeneous DE, then so is any linear combination $C_1 y_1 + C_2 y_2$.
  - **(general solution of a homogeneous linear DE)** For any homogeneous linear DE of order n, there are n solutions $y_1, y_2, \ldots, y_n$ such that the general solution is $y(x) = C_1 y_1(x) + \ldots + C_n y_n(x)$.

(general solution of an inhomogeneous linear DE) The general solution of any inhomogeneous linear DE of order n is of the form

$$y(x) = y_p(x) + C_1 y_1(x) + \ldots + C_n y_n(x),$$

where y_p is any solution (called a **particular solution**) and $C_1 y_1(x) + \ldots + C_n y_n(x)$ is the general solution of the corresponding homogeneous DE.

Why? Suppose we have a single solution y_p (called a **particular solution**) of the inhomogeneous DE $Ly = Q(x)$.
Let y_h be the general solution of the homogeneous DE $Ly = 0$.
Then $L(y_p + y_h) = \underbrace{Ly_p}_{=Q(x)} + \underbrace{Ly_h}_{=0} = Q(x)$.
In other words, $y_p + y_h$ solves $Ly = Q(x)$ as well. Indeed it must be the general solution (note that it has the appropriate number of degrees of freedom).

Comment. If $y_p^{(1)}$ and $y_p^{(2)}$ are two solutions of $Ly = Q(x)$. What can you say about $y_p^{(1)} - y_p^{(2)}$?

[This difference should solve the homogeneous DE $Ly = 0$. Indeed, $L\big(y_p^{(1)} - y_p^{(2)}\big) = \underbrace{Ly_p^{(1)}}_{=Q(x)} - \underbrace{Ly_p^{(2)}}_{=Q(x)} = 0$.]

Example 76. (preview) Determine the general solution of $y'' + 4y = 12x$. *Hint*: $3x$ is a solution.

Solution. Here, $p(D) = D^2 + 4$. Because of the hint, we know that a particular solution is $y_p = 3x$.
The homogeneous DE $p(D)y = 0$ has solutions $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$. [Make sure this is clear!]
Therefore, the general solution to the original DE is $y_p + C_1 y_1 + C_2 y_2 = 3x + C_1 \cos(2x) + C_2 \sin(2x)$.

Next. How to find the particular solution $y_p = 3x$ ourselves.

The method of undetermined coefficients

The method of undetermined coefficients allows us to solve any inhomogeneous linear DE $Ly = Q(x)$ with constant coefficients if $Q(x)$ is a polynomial times an exponential (or a linear combination of such terms).

More precisely, $Q(x)$ needs to be a solution of a homogeneous linear DE with constant coefficients.

Example 77. Determine the general solution of $y'' + 4y = 12x$.

Solution. Here, $p(D) = D^2 + 4$, which has roots $\pm 2i$.
Hence, the general solution is $y(x) = y_p(x) + C_1 \cos(2x) + C_2 \sin(2x)$. It remains to find a particular solution y_p.
Noting that $D^2 \cdot (12x) = 0$, we apply D^2 to both sides of the DE.
We get $D^2(D^2 + 4) \cdot y = 0$, which is a homogeneous linear DE! Its general solution is $C_1 + C_2 x + C_3 \cos(2x) + C_4 \sin(2x)$. In particular, y_p is of this form for some choice of $C_1, \ldots, C_4$.
It simplifies our life to note that there has to be a particular solution of the simpler form $y_p = C_1 + C_2 x$.
[Why?! Because we know that $C_3 \cos(2x) + C_4 \sin(2x)$ can be added to any particular solution.]
It only remains to find appropriate values C_1, C_2 such that $y_p'' + 4y_p = 12x$. Since $y_p'' + 4y_p = 4C_1 + 4C_2 x$, comparing coefficients yields $4C_1 = 0$ and $4C_2 = 12$, so that $C_1 = 0$ and $C_2 = 3$. In other words, $y_p = 3x$.
Therefore, the general solution to the original DE is $y(x) = 3x + C_1 \cos(2x) + C_2 \sin(2x)$.

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